# Abacus-histories and the combinatorics of creation operators

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July 30, 2020

**Abstract**

Creation operators act on symmetric functions to build Schur functions, Hall–Littlewood polynomials, and related symmetric functions one row at a time. Haglund, Morse, Zabrocki, and others have studied more general symmetric functions $H_\alpha$, $C_\alpha$, and $B_\alpha$ obtained by applying any sequence of creation operators to 1. We develop new combinatorial models for the Schur expansions of these and related symmetric functions using objects called abacus-histories. These formulas arise by chaining together smaller abacus-histories that encode the effect of an individual creation operator on a given Schur function. We give a similar treatment for operators such as multiplication by $h_m$, $h_m^\perp$, $\omega$, etc., which serve as building blocks to construct the creation operators. We use involutions on abacus-histories to give bijective proofs of properties of the Bernstein creation operator and Hall–Littlewood polynomials indexed by three-row partitions.

**Keywords:** Hall–Littlewood polynomials; Bernstein operators; Jing operators; HMZ operators; creation operators; Schur functions; semistandard tableaux; abaci; abacus histories; lattice paths.

## 1 Introduction

Creation operators are an important technical tool in the study of the Schur polynomials $s_\mu$, the Hall–Littlewood polynomials $H_\mu$, and related symmetric functions. Let $\Lambda$ denote the ring of symmetric functions with coefficients in the field $F = \mathbb{Q}(q)$, where $q$ is a formal variable. For each integer $b$, the *Bernstein creation operator* $\mathbb{S}_b$ is an $F$-linear operator on $\Lambda$. These operators "create" the Schur symmetric functions, one row at a time, in the following sense. Given any integer partition $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_L)$, we have

$$s_\mu = \mathbb{S}_{\mu_1} \circ \mathbb{S}_{\mu_2} \circ \cdots \circ \mathbb{S}_{\mu_L}(1). \tag{1}$$

Similarly, the *Jing creation operators* [6] are $F$-linear operators $\mathbb{H}_b$ on $\Lambda$ that create the Hall–Littlewood symmetric functions $H_\mu$ [15, Chpt. III]. Specifically, for any integer partition $\mu$,

$$H_\mu = \mathbb{H}_{\mu_1} \circ \mathbb{H}_{\mu_2} \circ \cdots \circ \mathbb{H}_{\mu_L}(1). \tag{2}$$

*This work was supported by a grant from the Simons Foundation/SFARI (#633564, N.A.L.).
†This work was supported by a grant from the Simons Foundation/SFARI (#429570, G.S.W.).

Garsia, Haglund, Morse, Xin, and Zabrocki [3, 4] have studied variations of the Jing creation operators, denoted $\mathbb{C}_b$ and $\mathbb{B}_b$, that play a crucial role in the study of $q,t$-Catalan numbers, diagonal harmonics modules, and the Bergeron–Garsia nabla operator. Replacing each $\mathbb{H}_{\mu_i}$ in (2) by $\mathbb{C}_{\mu_i}$ or $\mathbb{B}_{\mu_i}$ produces symmetric functions that are closely related to Hall–Littlewood polynomials. More generally, we can consider operators indexed by arbitrary compositions rather than restricting to partitions. For any finite sequence of integers $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_L)$, we can define symmetric functions

$$S_\alpha = \mathbb{S}_{\alpha_1} \circ \mathbb{S}_{\alpha_2} \circ \cdots \circ \mathbb{S}_{\alpha_L}(1); \tag{3}$$

$$H_\alpha = \mathbb{H}_{\alpha_1} \circ \mathbb{H}_{\alpha_2} \circ \cdots \circ \mathbb{H}_{\alpha_L}(1); \tag{4}$$

$$C_\alpha = \mathbb{C}_{\alpha_1} \circ \mathbb{C}_{\alpha_2} \circ \cdots \circ \mathbb{C}_{\alpha_L}(1); \tag{5}$$

$$B_\alpha = \mathbb{B}_{\alpha_L} \circ \mathbb{B}_{\alpha_{L-1}} \circ \cdots \circ \mathbb{B}_{\alpha_1}(1). \tag{6}$$

On one hand, as we explain in Section 2.2, each $S_\alpha$ is either 0 or $\pm s_\mu$ for some partition $\mu$, where $\mu$ can be found from $\alpha$ by repeated use of the *commutation rule*

$$\mathbb{S}_m \circ \mathbb{S}_n = -\,\mathbb{S}_{n-1} \circ \mathbb{S}_{m+1} \qquad (m, n \in \mathbb{Z}). \tag{7}$$

On the other hand, the $H_\alpha$, $C_\alpha$, and $B_\alpha$ are more complicated symmetric functions that may be regarded as generalized Hall–Littlewood polynomials. For general $\alpha$, the Schur coefficients of these symmetric functions are polynomials in $q$ (possibly multiplied by a fixed negative power of $q$) containing a mixture of positive and negative coefficients.

The primary goal of this paper is to develop explicit combinatorial formulas for the Schur expansions of $H_\alpha$, $C_\alpha$, and $B_\alpha$ based on signed, weighted collections of non-intersecting lattice paths called *abacus-histories*. Along the way, we develop concrete formulas giving the Schur expansion of the image of any Schur function under a single operator $\mathbb{S}_b$, $\mathbb{H}_b$, $\mathbb{C}_b$, $\mathbb{B}_b$, or any finite composition of such operators. We also give a similar treatment for some simpler operators such as $\omega$, multiplication by $h_b$, $h_b^\perp$, etc., which serve as building blocks for constructing the more elaborate creation operators.

Some related work appears in a paper by Jeff Remmel and Meesue Yoo [17]. Our approach features two key innovations leading to new and explicit combinatorial formulas. First, we use *abacus diagrams* rather than *Ferrers diagrams* as a means of visualizing the indexing partition $\mu$ for a Schur function $s_\mu$. This lets us record a particular Schur coefficient using a one-dimensional picture instead of a two-dimensional picture. Second, we utilize the second dimension of our picture to show the *evolution of the abacus over time* as various operators are applied to our initial Schur function. We thereby generate collections of non-intersecting lattice paths (abacus-histories) that represent the individual terms in the Schur expansion of the desired symmetric function. In some instances, we can define involutions on abacus-histories that cancel out negative objects, thereby proving Schur-positivity or related identities.

The rest of this paper is organized as follows. Section 2 reviews the needed background on symmetric functions and covers definitions and algebraic properties of various creation operators. Section 3 develops combinatorial versions of the creation operators, showing how to implement each operator by acting on an abacus diagram for one or two time steps. We use this combinatorics to reprove (from first principles) some creation operator identities such as (1) and (7). Section 4 iterates our constructions to develop abacus-history formulas for the Schur expansions of $H_\alpha$, $C_\alpha$, $B_\alpha$, and related symmetric functions. As a sample application of this technology, Section 4.2 gives an elementary combinatorial proof of the Schur-positivity of three-row Hall–Littlewood polynomials, leading to a simple formula for the Schur coefficients of these objects. We conclude by presenting some open problems and directions for further work.

# 2 Algebraic Development of Creation Operators

We assume readers are familiar with basic background material on symmetric functions, including definitions and facts concerning integer partitions, the elementary symmetric functions $e_k$, the complete homogeneous symmetric functions $h_k$, the Schur symmetric functions $s_\mu$, the involution $\omega$, and the Hall scalar product $\langle \cdot, \cdot \rangle$ on $\Lambda$. In particular, the Schur functions $s_\mu$ (with $\mu$ ranging over all integer partitions) form an orthonormal basis of $\Lambda$ relative to the Hall scalar product, and $\omega$ is an involution, ring isomorphism, and isometry on $\Lambda$ sending each $s_\mu$ to $s_{\mu'}$. (See standard texts on symmetric functions such as [13, 15, 18] for more information.)

Our initial definitions of the creation operators (following [3, 4]) utilize *plethystic notation*, but readers need not have any detailed prior knowledge of plethystic notation to understand this paper. In fact, one of our goals here is to offer an alternative, highly concrete and combinatorial treatment of creation operators to complement the plethystic computations that appear in much of the existing literature on this topic. Thus, each plethystic definition is immediately followed by an equivalent algebraic definition not using plethystic notation. Familiarity with plethystic notation is required in only one (optional) section that proves the equivalence of the two definitions. The paper [14] has a detailed exposition of plethystic notation containing all facts needed here. Later in the paper, we develop completely combinatorial definitions of creation operators (and related operators) in terms of abacus-histories.

## 2.1 Multiplication Operators and their Adjoints

Recall that $F$ is the field $\mathbb{Q}(q)$, and $\Lambda$ is the $F$-algebra of symmetric functions with coefficients in $F$. For any symmetric function $f \in \Lambda$, define the linear operator $M_f : \Lambda \to \Lambda$ to be *multiplication by* $f$:

$$M_f(P) = fP \qquad (P \in \Lambda). \tag{8}$$

We frequently take $f$ to be $h_c$ (the complete homogeneous symmetric function) or $e_c$ (the elementary symmetric function).

The Pieri Rules [13, Sec. 9.11] show how $M_{h_c}$ and $M_{e_c}$ act on the Schur basis. First, let $\mathrm{HS}(c)$ be the set of all skew shapes $\lambda/\mu$ consisting of a horizontal strip of $c$ cells. For all partitions $\mu$,

$$M_{h_c}(s_\mu) = h_c s_\mu = \sum_{\lambda:\ \lambda/\mu \in \mathrm{HS}(c)} s_\lambda. \tag{9}$$

Pictorially, we apply $M_{h_c}$ to $s_\mu$ by adding a horizontal strip of size $c$ to the Ferrers diagram of $\mu$ in all possible ways and summing the Schur functions indexed by the new diagrams.

Second, let $\mathrm{VS}(c)$ be the set of all skew shapes $\lambda/\mu$ consisting of a vertical strip of $c$ cells. For all partitions $\mu$,

$$M_{e_c}(s_\mu) = e_c s_\mu = \sum_{\lambda:\ \lambda/\mu \in \mathrm{VS}(c)} s_\lambda. \tag{10}$$

This time, we compute $M_{e_c}(s_\mu)$ by adding a vertical strip of size $c$ to the diagram of $\mu$ in all possible ways and summing the resulting Schur functions.

For any linear operator $G$ on $\Lambda$, let $G^{\perp}$ denote the operator on $\Lambda$ that is *adjoint* to $G$ relative to the Hall scalar product. So, $G^{\perp}$ is the unique linear map on $\Lambda$ satisfying

$$\langle G^{\perp}(P), Q\rangle = \langle P, G(Q)\rangle \quad \text{for all } P, Q \in \Lambda. \tag{11}$$

When $G$ is a multiplication operator $M_f$, we define $f^{\perp} = (M_f)^{\perp}$ for brevity. Thus,

$$\langle f^{\perp}(P), Q\rangle = \langle P, fQ\rangle \quad \text{for all } f, P, Q \in \Lambda. \tag{12}$$

We mostly use $h_c^{\perp}$ and $e_c^{\perp}$ acting on the Schur basis. Since the Schur basis is orthonormal relative to the Hall scalar product, it follows from (12) and (9) that

$$h_c^{\perp}(s_\mu) = \sum_{\nu:\ \mu/\nu \in \mathrm{HS}(c)} s_\nu. \tag{13}$$

In other words, $h_c^{\perp}$ acts on $s_\mu$ by *removing* a horizontal $c$-strip from the Ferrers diagram of $\mu$ in all possible ways and summing the resulting Schur functions. Similarly,

$$e_c^{\perp}(s_\mu) = \sum_{\nu:\ \mu/\nu \in \mathrm{VS}(c)} s_\nu. \tag{14}$$

So $e_c^{\perp}$ acts on $s_\mu$ by removing a vertical $c$-strip from the Ferrers diagram of $\mu$ in all possible ways and summing the resulting Schur functions.

As a convention, when $c$ is a negative integer, we define the operators $M_{h_c}$, $M_{e_c}$, $h_c^{\perp}$, and $e_c^{\perp}$ to be the zero operator.

### 2.2 Bernstein's Creation Operators $\mathbb{S}_m$

As in [4, pg. 834], we give a plethystic formula defining the Bernstein creation operators $\mathbb{S}_m$. More information on these operators (which can be combined into a single operator denoted $\mathbb{S}$ or $\Gamma(z)$) appears in earlier works by Thibon et al. [19, 20, 21]. For any integer $m$ and any symmetric function $P$, set

$$\mathbb{S}_m(P) = \left\{ P\left[X - \frac{1}{z}\right] \sum_{k=0}^{\infty} h_k z^k \right\}\Bigg|_{z^m}. \tag{15}$$

To explain this formula briefly: we first compute $P[X-(1/z)]$ by expressing $P$ (uniquely) as a polynomial in the power-sum symmetric functions $p_n$ with coefficients in $F$, then replacing each $p_n$ by $p_n - (1/z^n)$. Next we multiply by the formal power series $\sum_{k\geq 0} h_k z^k$ to obtain a formal Laurent series in $z$ with coefficients in $\Lambda$. Taking the coefficient of $z^m$ in this series gives us $\mathbb{S}_m(P)$.

Here is an equivalent algebraic definition of $\mathbb{S}_m$ not using plethystic notation:

$$\mathbb{S}_m = \sum_{c=0}^{\infty} (-1)^c M_{h_{m+c}} \circ e_c^{\perp}. \tag{16}$$

(This definition appears in [15, Ex. 29, pp. 95–97], but Macdonald uses the notation $B_m$ for our $\mathbb{S}_m$. We use the notation $\mathbb{S}_m$ from [4] to avoid confusion with the operator $\mathbb{B}_m$ below.) We prove the equivalence of definitions (15) and (16) in Section 5.

| | | |
|---|---|---|
| $\mathbb{S}_m(s_\mu) = +s_{(m,88844221)}$ for all $m \geq 8$, | $\mathbb{S}_4(s_\mu) = -s_{(777744221)}$, | $\mathbb{S}_3(s_\mu) = -s_{(777644221)}$, |
| $\mathbb{S}_2(s_\mu) = -s_{(777544221)}$, | $\mathbb{S}_1(s_\mu) = -s_{(777444221)}$, | $\mathbb{S}_{-2}(s_\mu) = -s_{(777333221)}$, |
| $\mathbb{S}_{-3}(s_\mu) = -s_{(777332221)}$, | $\mathbb{S}_{-6}(s_\mu) = -s_{(777331111)}$, | $\mathbb{S}_{-8}(s_\mu) = +s_{(777331100)}$, |
| $\mathbb{S}_m(s_\mu) = 0$ for all other $m \in \mathbb{Z}$. | | |

Table 1: Values of $\mathbb{S}_m(s_{(88844221)})$ for all $m \in \mathbb{Z}$.

By combining the Pieri formulas and dual Pieri formulas discussed above, we can give a combinatorial prescription for computing $\mathbb{S}_m(s_\mu)$ based on Ferrers diagrams. Starting with the diagram of $\mu$, do the following steps in all possible ways. First choose a nonnegative integer $c$. Then remove a vertical strip of $c$ cells from $\mu$ to get some shape $\nu$. Then add a horizontal strip of $m+c$ cells to $\nu$ to get a new shape $\lambda$. Record $(-1)^c s_\lambda$ as one of the terms in $\mathbb{S}_m(s_\mu)$.

Now, there is a much simpler way of computing $\mathbb{S}_m(s_\mu)$ based on formulas (1), (3), and (7). Given a partition $\mu = (\mu_1 \geq \cdots \geq \mu_L)$ and integer $m$, start with the list $(m, \mu_1, \ldots, \mu_L)$. If $m \geq \mu_1$, then output the Schur function indexed by this new list. Otherwise, repeatedly perform the following steps on the list (with infinitely many zero parts appended). Initialize a sign variable $\epsilon = +1$. Look for the unique ascent $a < b$ in the current list. If $b = a+1$, then return zero as the answer. Otherwise replace the sublist $(a, b)$ by $(b-1, a+1)$, replace $\epsilon$ by $-\epsilon$, and continue. We eventually return zero or obtain a weakly decreasing list of nonnegative integers. In the latter case, return $\epsilon$ times the Schur function indexed by this list.

This algorithm is a version of Littlewood's method for straightening Jacobi–Trudi determinants [11]. Given any list of integers $\alpha = (\alpha_1, \ldots, \alpha_L)$, let $D(\alpha)$ be the determinant of the $L \times L$ matrix with $i, j$-entry $h_{\alpha_i+j-i}$. For an integer partition $\mu$, $D(\mu)$ is the Schur function $s_\mu$ by the Jacobi–Trudi formula. For any $\alpha$, $D(\alpha)$ is either 0 or $\pm s_\nu$ for some partition $\nu$. We can find $\nu$ by repeatedly interchanging rows $i$ and $i+1$ of the matrix where $\alpha_i < \alpha_{i+1}$. Each such interchange causes a sign change and replaces parts $\alpha_i$ and $\alpha_{i+1}$ in $\alpha$ by $\alpha_{i+1}-1$ and $\alpha_i+1$, respectively. Comparing to the previous paragraph, we see that $\mathbb{S}_m(s_\mu)$ is none other than $D(m, \mu_1, \ldots, \mu_L)$.

**Example 1.** Given $\mu = (8, 8, 8, 4, 4, 2, 2, 1)$ and $m = -2$, we compute

$$\begin{aligned}
\mathbb{S}_{-2}(s_\mu) &= S_{(-2,8,8,8,4,4,2,2,1)} = -S_{(7,-1,8,8,4,4,2,2,1)} = +S_{(7,7,0,8,4,4,2,2,1)} \\
&= -S_{(7,7,7,1,4,4,2,2,1)} = +S_{(7,7,7,3,2,4,2,2,1)} = -S_{(7,7,7,3,3,3,2,2,1)} \\
&= -s_{(777333221)}.
\end{aligned}$$

By similar calculations, we find the values of $\mathbb{S}_m(s_\mu)$ shown in Table 1.

It is not obvious that the two methods we have described for computing $\mathbb{S}_m(s_\mu)$ always give the same result. We prove this fact later using abacus-histories, and we will also give direct combinatorial proofs of (1) and (7).

### 2.3 Jing's Creation Operators $\mathbb{H}_m$

Here is a plethystic definition of Jing's creation operators $\mathbb{H}_m$. As in [4, (2.2)], for all $m \in \mathbb{Z}$ and $P \in \Lambda$, let

$$\mathbb{H}_m(P) = \left\{ P\left[X + \frac{q-1}{z}\right] \sum_{k=0}^{\infty} h_k z^k \right\}\Bigg|_{z^m}. \tag{17}$$

In this case, $P[X + (q-1)/z]$ denotes the image of $P$ under the plethystic substitution sending each $p_n$ to $p_n + (q^n - 1)/z^n$.

Alternatively, we could define

$$\mathbb{H}_m = \sum_{c \geq 0} q^c \, \mathbb{S}_{m+c} \circ h_c^{\perp}. \tag{18}$$

We prove the equivalence of definitions (17) and (18) in Section 5.

We can compute $\mathbb{H}_m(s_\mu)$ via Ferrers diagrams as follows. Starting with the diagram of $\mu$, do the following steps in all possible ways. First choose a nonnegative integer $c$. Then remove a horizontal strip of $c$ cells from $\mu$ to get some shape $\nu$. Compute $\mathbb{S}_{m+c}(\nu)$ as described earlier to obtain zero or a signed Schur function $\pm s_\lambda$. Record $q^c$ times the answer as one of the terms in the Schur expansion of $\mathbb{H}_m(s_\mu)$. By iterating this description, it is evident that for every list of integers $\alpha$, the Schur coefficients of $H_\alpha$ are polynomials in $q$ with integer coefficients. Jing [6] proved that when $\alpha$ is a partition $\mu$, $H_\mu = \mathbb{H}_{\mu_1} \circ \cdots \circ \mathbb{H}_{\mu_L}(1)$ is none other than the Hall–Littlewood symmetric function indexed by $\mu$.

### 2.4 The Creation Operators $\mathbb{C}_m$ and $\mathbb{B}_m$

We use [4, Remark 3.7, pg. 835] as our plethystic definition of the creation operator $\mathbb{C}_m$. For $m \in \mathbb{Z}$ and $P \in \Lambda$, let

$$\mathbb{C}_m(P) = \left\{ (-q^{-1})^{m-1} P\left[X + \frac{q^{-1}-1}{z}\right] \sum_{k=0}^{\infty} h_k z^k \right\}\Bigg|_{z^m}. \tag{19}$$

This operator is a variation of $\mathbb{H}_m$ obtained by replacing $q$ by $1/q$ in $\mathbb{H}_m$, and then multiplying the output by a global factor $(-1/q)^{m-1}$. So (18) leads at once to the following alternative definition of $\mathbb{C}_m$:

$$\mathbb{C}_m = (-q^{-1})^{m-1} \sum_{c \geq 0} q^{-c} \, \mathbb{S}_{m+c} \circ h_c^{\perp}. \tag{20}$$

Proposition 3.6 of [4] proves an inverse version of this identity, namely

$$\mathbb{S}_m = (-q)^{m-1} \sum_{i \geq 0} \mathbb{C}_{m+i} \circ e_i^{\perp}.$$

Creation operators satisfy some useful commutation relations. For example, Proposition 3.2 of [4] shows that for $m, n \in \mathbb{Z}$,

$$q \, \mathbb{C}_m \circ \mathbb{C}_n - \mathbb{C}_{m+1} \circ \mathbb{C}_{n-1} = \mathbb{C}_n \circ \mathbb{C}_m - q \, \mathbb{C}_{n-1} \circ \mathbb{C}_{m+1},$$

and in particular $q \, \mathbb{C}_m \circ \mathbb{C}_{m+1} = \mathbb{C}_{m+1} \circ \mathbb{C}_m$. Analogous relations for $\mathbb{H}_m$ were proved much earlier by Jing (see (1.1) in [6] or (0.18) in [7]).

Finally, we define the creation operator $\mathbb{B}_m$ by conjugating $\mathbb{H}_m$ by $\omega$ (see [4, pg. 829]):

$$\mathbb{B}_m = \omega \circ \mathbb{H}_m \circ \omega. \tag{21}$$

Recall that $\omega$ is the linear operator on $\Lambda$ sending each Schur function $s_\lambda$ to $s_{\lambda'}$, where $\lambda'$ is the partition conjugate to $\lambda$ obtained by transposing the Ferrers diagram of $\lambda$. Proposition 3.5 of [4] shows that for $m+n>0$, $\mathbb{B}_n \circ \mathbb{C}_m = q\,\mathbb{C}_m \circ \mathbb{B}_n$.

## 2.5 Algebraic Rules for Conjugation by $\omega$

Let $\mathscr{C}_\omega$ denote conjugation by $\omega$, which sends any operator $G$ on $\Lambda$ to $\mathscr{C}_\omega(G) = \omega \circ G \circ \omega$. We now give some useful identities involving $\mathscr{C}_\omega$. First,

$$\mathscr{C}_\omega(M_f) = M_{\omega(f)} \quad \text{for all } f \in \Lambda. \tag{22}$$

To check this, recall that $\omega$ is a ring homomorphism on $\Lambda$ and an involution ($\omega \circ \omega = \mathrm{id}$). So for any $P \in \Lambda$,

$$\mathscr{C}_\omega(M_f)(P) = \omega(M_f(\omega(P))) = \omega(f \cdot \omega(P)) = \omega(f) \cdot \omega(\omega(P)) = \omega(f) \cdot P = M_{\omega(f)}(P).$$

Second, $\omega^\perp = \omega$. This follows since $\omega$ is an involution and an isometry (relative to the Hall scalar product), which means that for all $P, Q \in \Lambda$, $\langle \omega(P), Q\rangle = \langle P, \omega(Q)\rangle = \langle \omega^\perp(P), Q\rangle$.

Third,

$$\mathscr{C}_\omega(f^\perp) = (\omega(f))^\perp \quad \text{for all } f \in \Lambda. \tag{23}$$

To see this, we use the first two facts and the adjoint property $(F \circ G)^\perp = G^\perp \circ F^\perp$ to compute:

$$\mathscr{C}_\omega(f^\perp) = \omega \circ (M_f)^\perp \circ \omega = \omega^\perp \circ (M_f)^\perp \circ \omega^\perp = (\omega \circ M_f \circ \omega)^\perp = (M_{\omega(f)})^\perp = (\omega(f))^\perp.$$

Fourth, using (16) and $\mathscr{C}_\omega(F \circ G) = \mathscr{C}_\omega(F) \circ \mathscr{C}_\omega(G)$, we find that

$$\mathscr{C}_\omega(\mathbb{S}_m) = \sum_{c \geq 0} (-1)^c \mathscr{C}_\omega(M_{h_{m+c}}) \circ \mathscr{C}_\omega(e_c^\perp) = \sum_{c \geq 0} (-1)^c M_{e_{m+c}} \circ h_c^\perp.$$

Fifth, using this result and (18), we get

$$\mathbb{B}_m = \mathscr{C}_\omega(\mathbb{H}_m) = \sum_{d \geq 0} q^d \mathscr{C}_\omega(\mathbb{S}_{m+d}) \circ \mathscr{C}_\omega(h_d^\perp) = \sum_{c \geq 0} \sum_{d \geq 0} q^d (-1)^c M_{e_{m+d+c}} \circ h_c^\perp \circ e_d^\perp.$$

So, we can compute $\mathbb{B}_m(s_\mu)$ via Ferrers diagrams as follows. Starting with the diagram of $\mu$, do the following steps in all possible ways. First, choose integers $c, d \geq 0$. Remove a vertical strip of $d$ cells from $\mu$, then remove a horizontal strip of $c$ cells, then add a vertical strip of $m+d+c$ cells. Record $q^d(-1)^c$ times the Schur function indexed by the new shape as one of the terms in the Schur expansion of $\mathbb{B}_m(s_\mu)$. Later, we use abacus-histories to find a more efficient combinatorial rule for computing this Schur expansion.

# 3 Combinatorial Development of Creation Operators

This section develops combinatorial formulas for the Schur expansions of $G(s_\mu)$, where $s_\mu$ is any Schur function and $G$ is one of the operators $M_{h_m}$, $h_m^\perp$, $M_{e_m}$, $e_m^\perp$, $\omega$, $\mathbb{S}_m$, $\mathbb{H}_m$, $\mathbb{C}_m$, or $\mathbb{B}_m$. These formulas are based on the *abacus model* for representing integer partitions. James and Kerber [5] introduced abaci to prove facts about $k$-cores and $k$-quotients of integer partitions. Abaci with labeled beads can be used to prove many fundamental facts about Schur functions, including the Pieri Rules for expanding $s_\mu h_k$ and $s_\mu e_k$ and the Littlewood–Richardson Rule [12]. In our work here, it suffices to consider unlabeled abaci. We introduce the new ingredient of tracking the evolution of the abacus over time to model compositions of operators applied to a given Schur function. This leads to new combinatorial objects called *abacus-histories* that model the Schur expansions of $H_\alpha$, $C_\alpha$, $B_\alpha$, and other symmetric functions built by composing creation operators.

## 3.1 The Abacus Model

First we review the correspondence between partitions and abaci. Suppose $N > 0$ is fixed and $\mu = (\mu_1, \mu_2, \ldots, \mu_N)$ is an integer partition (weakly decreasing sequence) consisting of $N$ nonnegative parts. Let $\delta(N) = (N-1, N-2, \ldots, 2, 1, 0)$. The map sending $\mu$ to $\mu + \delta(N) = (\mu_1 + N - 1, \mu_2 + N - 2, \ldots, \mu_N)$ is a bijection from the set of weakly decreasing sequences of $N$ nonnegative integers onto the set of strictly decreasing sequences of $N$ nonnegative integers. We visualize the sequence $\mu + \delta(N)$ by drawing an abacus with positions numbered $0, 1, 2, \ldots$, and placing a bead in position $\mu_i + N - i$ for $1 \leq i \leq N$. We use $\mu + \delta(N)$ rather than $\mu$ because each position can hold at most one bead. To formalize this concept, we define an *$N$-bead abacus* to be a word $w = w_0 w_1 w_2 \cdots$ with all $w_i \in \{0, 1\}$ and $w_i = 1$ for exactly $N$ indices $i$. Here $w_i = 1$ means the abacus has a bead in position $i$, while $w_i = 0$ means the abacus has a gap in position $i$. For example, if $N = 10$ and $\mu = (8, 8, 8, 4, 4, 2, 2, 1, 0, 0)$, the associated abacus is

$$w = 1101011001100011100000\cdots. \tag{24}$$

In the theory of symmetric functions, we usually identify two partitions that differ only by adding or deleting zero parts. In fact, it is often most convenient to regard an integer partition as an infinite weakly decreasing sequence ending in infinitely many zeroes. To model such a sequence $\mu$ as an abacus, we use a doubly-infinite word $w = (w_i : i \in \mathbb{Z})$ such that $w_i = 1$ for all $i < 0$, $w_0 = 0$, and $w_i = 1$ for only finitely many indices $i \geq 0$. The nonzero parts of $\mu$ can be recovered from the abacus $w$ by counting the number of gaps to the left of each bead on the positive side of the abacus. The convention of putting the leftmost gap at position 0 is not essential; any left-shift or right-shift of the word $w$ leads to the same combinatorics.

We can also construct the abacus for a partition $\mu$ by following the frontier of the Ferrers diagram of $\mu$. As illustrated in Figure 1, the frontier consists of infinitely many north steps (corresponding to the infinitely many zero parts at the end of the sequence $\mu$), followed by a sequence of east and north steps that follow the edge of the diagram, followed by infinitely many east steps at the top edge. Replacing each north step with a bead, replacing each east step with a gap, and declaring the first east step to have index 0 produces the doubly-infinite abacus associated with $\mu$. The singly-infinite abacus $w$ in (24) is the ten-bead version of the full abacus shown in Figure 1, obtained by discarding all beads to the left of the tenth bead from the right and shifting the origin to this location.

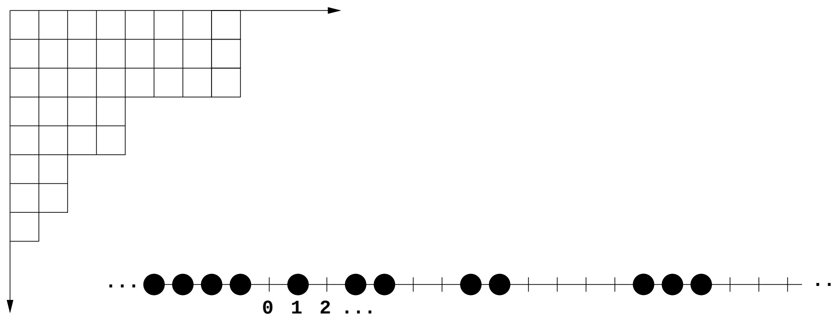


Figure 1: The Ferrers diagram of $\mu$ and the doubly-infinite abacus built from the frontier of $\mu$.

For convenience, we mostly use $N$-bead abaci in this paper, which leads to identities valid for symmetric polynomials in $N$ variables. However, for certain abacus moves to work, we must be sure to pad $\mu$ with enough zero parts (corresponding to beads at the far left end of the abacus) since these beads might participate in the move. This reflects the algebraic fact that some symmetric function identities are only true provided the number of variables is large enough.

### 3.2 Abacus Versions of $M_{h_m}$ and $h_m^\perp$

Now we describe how to compute $G(s_\mu)$ using abaci, for various operators $G$. Here and below, we represent the input $s_\mu$ as an $N$-bead abacus drawn in the top row of a diagram. Each operator $G$ acts by moving beads on the abacus according to certain rules, producing several possible new abaci that may be multiplied by signs or weights (powers of $q$). Each abacus stands for the Schur function indexed by the partition corresponding to the abacus. We make a diagram for each possible new abacus produced by $G$, where the output abacus appears in the second row (see the figures below for examples). Moving downward through successive rows represents the passage of time as various operators are applied to the initial abacus. This convention lets us use a two-dimensional picture to display the evolution of an abacus as a whole sequence of operators are performed. It is much more difficult to visualize such an operator sequence using Ferrers diagrams, particularly when some operators act by adding cells and others act by removing cells.

As our first example, consider the computation of $M_{h_m}(s_\mu) = h_m s_\mu$ using abaci. By the Pieri Rule (9), we know $h_m s_\mu$ is the sum of all $s_\lambda$ where $\lambda$ is obtained by adding a horizontal strip of $m$ cells to the diagram of $\mu$ in all possible ways. Using the correspondence between the frontier of $\mu$ and the abacus for $\mu$, it is routine to check that adding such a horizontal strip corresponds to moving various beads right a total of $m$ positions on the abacus. (See [12] or [13, Sec. 10.10] for more details.) A given bead may move more than once, but no bead may move into a position originally occupied by another bead.

To record this move in an abacus-history diagram, we start in row 1 with an $N$-bead abacus for $\mu$, where $\mu$ must end in at least one part equal to 0. We draw the beads as dots located at lattice points in row 1. Next, in all possible ways, we draw a total of $m$ east steps starting at various beads, then move every bead 1 step south to represent the passage of time. For example, Figure 2 shows the abacus-histories encoding the computation

$$M_{h_2}(s_{(3110)}) = s_{(5110)} + s_{(4210)} + s_{(4111)} + s_{(3310)} + s_{(3211)}.$$

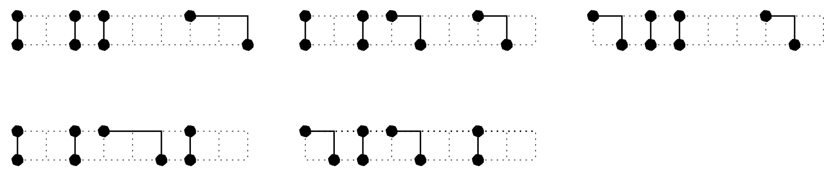

Figure 2: Computing $M_{h_2}(s_{(3110)})$ using abacus-histories.

Figure 3: Computing $h_2^\perp(s_{(332)})$ using abacus-histories.

Note that the extra zero part is needed to accommodate horizontal strips that use cells in the row below the last nonzero part of $\mu$.

Next consider how to compute $h_m^\perp(s_\mu)$. Recall that the answer is the sum of all $s_\nu$ where $\nu$ can be obtained by *removing* a horizontal strip of $m$ cells from the Ferrers diagram of $\mu$. To execute this action on an abacus, we move various beads *west* a total of $m$ positions, avoiding collisions with the original locations of the beads. Then we move every bead south one step to represent the passage of time. For example, Figure 3 shows the abacus-histories encoding the computation

$$h_2^\perp(s_{(332)}) = s_{(330)} + s_{(321)}.$$

Note that $\mu$ need not be padded with zero parts when using this rule.

### 3.3 Abacus Versions of $M_{e_m}$ and $e_m^\perp$

Now we describe abacus implementations of $M_{e_m}$ and $e_m^\perp$. Recall $M_{e_m}(s_\mu)$ is the sum of all $s_\lambda$ where $\lambda$ is obtained by adding a vertical strip of $m$ cells to the Ferrers diagram of $\mu$. Comparing the frontiers of $\mu$ and $\lambda$, we see that adding such a vertical strip corresponds to simultaneously moving $m$ distinct beads east one step each. Beads cannot collide on the new abacus, but a bead is allowed to move into a position vacated by another bead.

To record this move in an abacus-history diagram, start with an $N$-bead abacus for $\mu$ where $\mu$ is padded with at least $m$ zero parts. In all possible ways, pick a set of $m$ beads that each move southeast one step, while the remaining beads move south one step with no collisions. For example, Figure 4 shows the abacus-histories encoding the computation

$$M_{e_2}(s_{(41100)}) = s_{(41111)} + s_{(42110)} + s_{(51110)} + s_{(42200)} + s_{(52100)}.$$

Note that the $m$ zero parts are needed to accommodate a vertical strip that we might add below the last nonzero part of $\mu$.

Next, $e_m^\perp(s_\mu)$ is the sum of all $s_\nu$ where $\nu$ is obtained from the Ferrers diagram of $\mu$ by removing a vertical strip of $m$ cells. Starting with the abacus for $\mu$, we pick a set of $m$ beads that each move southwest one step, while the remaining beads move south one step with no collisions. For example, Figure 5 shows the abacus-histories encoding the computation

$$e_2^\perp(s_{(442)}) = s_{(431)} + s_{(332)}.$$

Note that $\mu$ need not be padded with zero parts when using this rule.

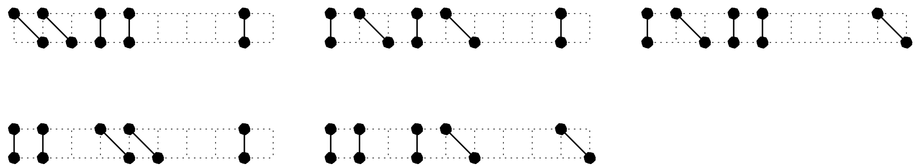

Figure 4: Computing $M_{e_2}(s_{(41100)})$ using abacus-histories.

Figure 5: Computing $e_2^\perp(s_{(442)})$ using abacus-histories.

### 3.4 Effects of $\omega$ and $\mathscr{C}_\omega$

We know $\omega(s_\mu) = s_{\mu'}$, where the Ferrers diagram of $\mu'$ is found by transposing the diagram of $\mu$. This transposition interchanges the roles of north and east steps on the frontier of $\mu$ and reverses the order of these steps. So, $\omega$ acts on the doubly-infinite abacus for $\mu$ by interchanging beads and gaps and reversing the abacus. Let us call this move an *abacus-flip*.

Now suppose we know a description of an operator $G$ in terms of moves on an abacus. Then $\mathscr{C}_\omega(G) = \omega \circ G \circ \omega$ acts on the abacus for $s_\mu$ by doing an abacus-flip, then doing the moves for $G$, then doing another abacus-flip. Therefore, we obtain a description of the operator $\mathscr{C}_\omega(G)$ from the given description of $G$ by interchanging the roles of beads and gaps, and interchanging the roles of east and west.

For example, consider $G = M_{h_m}$ and $\mathscr{C}_\omega(G) = M_{e_m}$. We know $G$ acts on the abacus for $s_\mu$ by moving some *beads* $m$ steps *east*, avoiding the original positions of all *beads*. Therefore, we can say that $\mathscr{C}_\omega(G)$ acts on the abacus for $s_\mu$ by moving some *gaps* $m$ steps *west* avoiding the original positions of all *gaps*. One may check that this description of $M_{e_m}$ (involving gap motions) is equivalent to the description given earlier (involving bead motions). When composing several operators to build bigger abacus-histories, it is often easier to work with descriptions that always move beads rather than gaps.

### 3.5 Abacus Version of $\mathbb{S}_m$

We develop an initial abacus-history implementation of the operator $\mathbb{S}_m$ based on formula (16), which will subsequently be simplified using a sign-reversing involution on abacus-histories. Given any partition $\mu$, we know by (16) that

$$\mathbb{S}_m(s_\mu) = \sum_{c \geq 0} (-1)^c h_{m+c} e_c^\perp(s_\mu).$$

To compute this with an abacus-history, start with the abacus for $\mu$ (padding $\mu$ with a zero part if needed) and perform the following steps in all possible ways. Choose an integer $c \geq 0$. In the first time step, apply $e_c^\perp$ by moving $c$ distinct beads one step southwest while the remaining beads move one step south with no collisions. In the second time step, apply $h_{m+c}$ by moving some beads a total of $m + c$ steps east, never moving into a position occupied by a bead at the start of this time step; then move all beads one step south. Record a term $(-1)^c$ times the

Schur function corresponding to the final abacus. For example, Figure 6 shows how to compute $\mathbb{S}_1(s_{(3110)})$ via abacus-histories. When $c = 0$, we obtain the three positive objects labeled A, B, C; when $c = 1$, we obtain the nine negative objects labeled D through L; and so on. Adding up the 23 signed Schur functions encoded by these abaci, there is massive cancellation leading to the answer $-s_{(2211)}$.

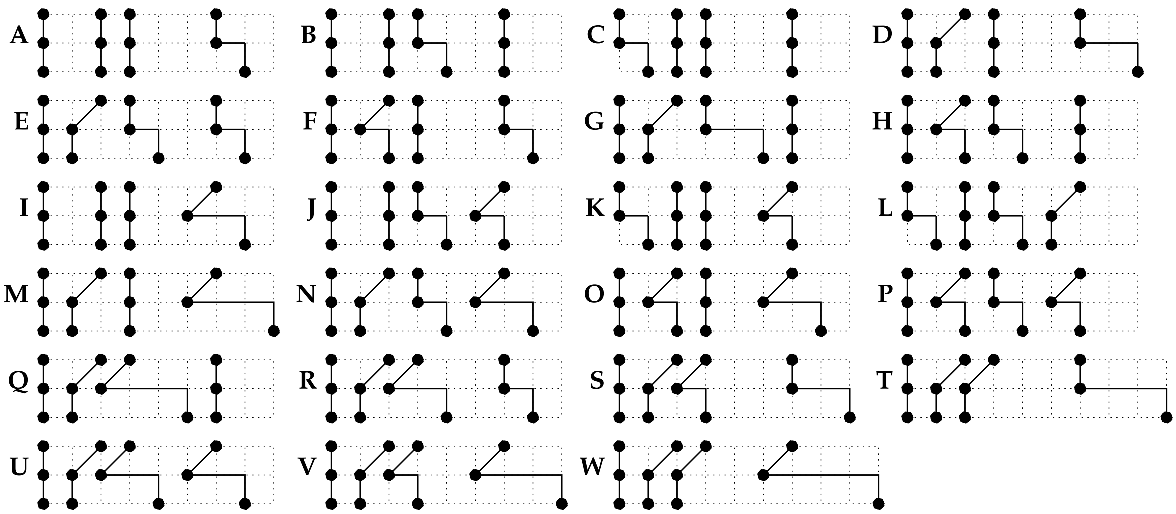


Figure 6: Initial computation of $\mathbb{S}_1(s_{(3110)})$ using abacus-histories.

We now introduce a sign-reversing involution on abacus-histories that explains the cancellation in the last example. Suppose an abacus-history appearing in the computation of $\mathbb{S}_m(s_\mu)$ contains a bead that moves southwest, then moves east $i > 0$ steps and then south, as shown on the left in Figure 7. By changing the first two steps from southwest-east to a single south step, the bead now moves as shown on the right in Figure 7, where the $e$ denotes a gap on the abacus that no bead visits during the second time interval. Conversely, if a bead initially moves south and then has a gap to its left that no bead visits, then we can replace this initial south step with a southwest step followed by an east step. These path modifications change $c$ by 1 and hence change the sign of the abacus-history, while preserving the requirement of taking $m + c$ total east steps in the second time interval. The involution acts on an abacus-history by scanning for the leftmost occurrence of one of the patterns in Figure 7 and replacing it with the other pattern. It is clear that doing the involution twice restores the original object. The fixed points of the involution (which could be negative) consist of all abacus-histories avoiding both patterns in Figure 7. For example, applying the involution to the objects in Figure 6 produces the following matches:

$$A \leftrightarrow F,\ B \leftrightarrow H,\ C \leftrightarrow K,\ D \leftrightarrow S,\ E \leftrightarrow R,\ G \leftrightarrow Q,\ I \leftrightarrow O,\ J \leftrightarrow P,\ M \leftrightarrow V,\ N \leftrightarrow U,\ T \leftrightarrow W.$$

We are left with the single negative fixed point $L$, confirming that $\mathbb{S}_1(s_{(3110)}) = -s_{(2211)}$.

We can now prove a formula for $\mathbb{S}_m(s_\mu)$ that interprets $\mathbb{S}_m$ as a **bead-creation operator** for abacus-histories. As consequences of this formula, we can finally justify equations (1), (7), and the technique for computing $\mathbb{S}_m(s_\mu)$ used in Example 1. To state the formula, we need some preliminary definitions. For any partition $\mu$, assign a *label* and a *sign* to each gap on the abacus

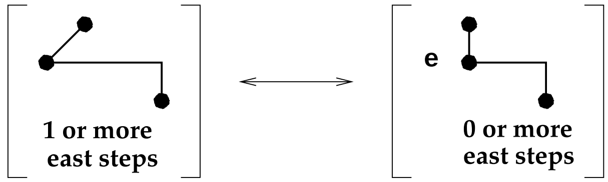


Figure 7: Cancellation move for abacus-histories appearing in $\mathbb{S}_m(s_\mu)$.

for $\mu$ as follows. Given a gap with $g$ gaps strictly to its left and $b$ beads to its right, let this gap have label $g-b$ and sign $(-1)^b$. Here is another way to compute the gap labels. The gap to the right of the rightmost bead on the abacus for $\mu$ has label $\mu_1$. Any gap $i$ positions to the right (resp. left) of this gap on the abacus has label $\mu_1+i$ (resp. $\mu_1-i$), as is readily checked.

**Theorem 2.** *For any partition $\mu$ and integer $m$, $\mathbb{S}_m(s_\mu)$ is 0 if no gap in the abacus for $\mu$ has label $m$. Otherwise, we compute $\mathbb{S}_m(s_\mu)$ by filling the unique gap labeled $m$ with a new bead, then multiplying the Schur function for the new abacus by the sign of this gap.*

*Proof.* First compute $\mathbb{S}_m(s_\mu)$ by generating a collection of signed abacus-histories, as described at the start of this section. Next apply the sign-reversing involution to cancel out pairs of objects. We must now analyze the structure of the fixed points that remain. All fixed points avoid occurrences of both patterns shown in Figure 7. From our abacus-history characterization of the action of $\mathbb{S}_m(s_\mu)$, there are only two other possible move patterns for a bead starting in the abacus for $s_\mu$. The first is for a bead to move south and then east zero or more east steps, but there must be another bead southwest of this bead's starting point (i.e., in the position marked $e$ on the right side of Figure 7). The second is for a bead to move southwest and then south with no intervening east steps. In the second case, we refer to this pair of steps as a *zig-down move* and say that the bead *zigs down*.

Given a fixed point, suppose there is a bead $Q$ on the input abacus that makes a zig-down move. On one hand, suppose there is a bead $P$ immediately to the left of $Q$ (i.e., with no gaps in between). Then $P$ must also zig down, since there is no room to do anything else. On the other hand, suppose $R$ is the next bead somewhere to the right of $Q$ (if any). There must be a vacancy immediately southwest of $R$'s initial position (since $Q$ zigs down). It follows that $R$ must also zig down to avoid the two forbidden patterns. Define a *block* of beads on an abacus to be a maximal set of beads with no gaps between any pair of them. Iterating our two preceding observations, we conclude that *if some bead zigs down in a fixed point, then all beads to its right and all beads to its left in its block must also zig down.*

Next consider a bead $S$ on the input abacus that has $g>0$ gaps to its right followed by another bead $T$ that does not zig down. On one hand, $S$ does not zig down (or else $T$ would too). On the other hand, for $T$ to avoid the second pattern in Figure 7, bead $S$ must move south, then $g$ steps east, then south. We now see that all fixed points for $\mathbb{S}_m(s_\mu)$ must have the following structure. There are zero or more beads at the right end that all zig down, starting with some bead $Q$ at the beginning of a block of beads. The motions of all remaining beads are uniquely determined (they move east as far as possible), except for the next bead $P$ to the left of $Q$. If beads $P$ and $Q$ are separated by $g>0$ gaps on the input abacus, then $P$ has the option of moving south, then $i$ steps east, then south, for any $i$ satisfying $0\leq i<g$. As a special case, if no beads zig down, then $P$ is the rightmost bead on the abacus, which can move $i$ steps east

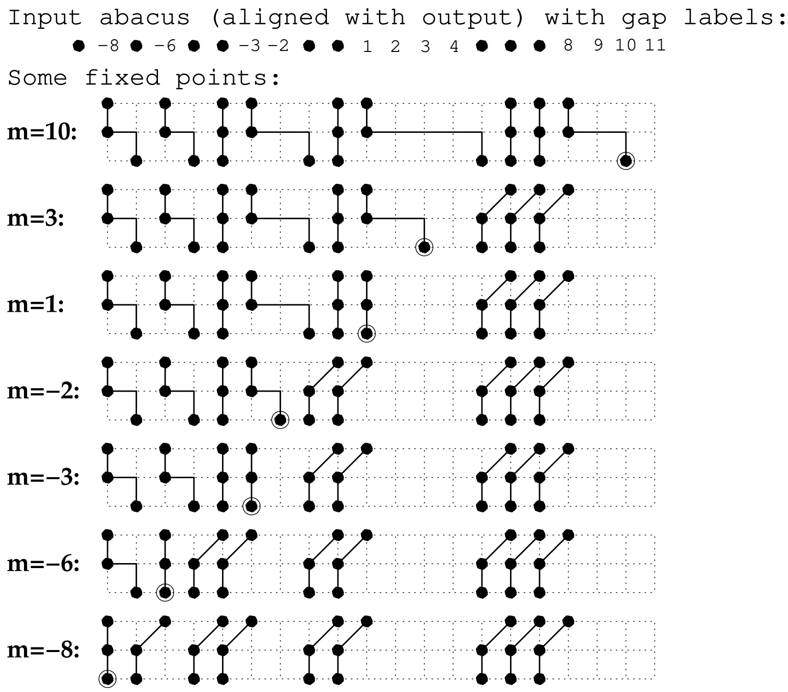


Figure 8: Fixed points in the computation of $\mathbb{S}_m(s_{(888442210)})$ for various values of $m$. Newly created beads are circled.

for any $i \geq 0$. For example, Figure 8 illustrates some of the fixed points for $\mathbb{S}_m(s_{(888442210)})$ for various choices of $m$ (compare to Example 1).

It is visually evident from this example that, for general $\mu$ and any fixed point of $\mathbb{S}_m(s_\mu)$, the output abacus arises from the input abacus by shifting every bead one position to the left (which corresponds to deleting a zero part from the end of $\mu$) and then filling one gap with a new bead. Suppose this gap has $g$ gaps strictly to its left and $b$ beads to its right on the abacus for $\mu$. The label of this gap is, by definition, $g - b$. To complete the proof, we need only confirm that $g - b = m$. By our characterization of fixed points, there are $b$ southwest steps in time interval 1 (since all beads to the right of this gap zig down). By (16) and our observations at the beginning of this section, these southwest steps arise from the action of $e_b^\perp$. As such, there must be $m + b$ east steps arising from the subsequent action of $h_{m+b}$. But, as illustrated in Figure 8, these east steps are in bijection with the gaps to the left of the new bead. So the number of gaps $g$ is $m + b$. It follows that $g - b = (m + b) - b = m$, as needed. □

Using the second method of computing gap labels, we see that for any partition $\mu$ and any integer $m \geq \mu_1$, $\mathbb{S}_m(s_\mu) = +s_{(m,\mu)}$. Iterating this result starting with $s_0 = 1$, we obtain (1). Similarly, formula (7) can be deduced quickly from Theorem 2 by the following abacus-based proof. The left side of (7) acts on $s_\mu$ by first creating a new bead in the gap labeled $n$, then creating a new bead in the gap now labeled $m$ (returning zero if either gap does not exist). One readily checks that creating a new bead in the gap labeled $n$ has the effect of decrementing all remaining gap labels. Thus, $\mathbb{S}_m \circ \mathbb{S}_n$ acts by filling the gap labeled $n$, then filling the gap *originally* labeled $m+1$, if these gaps exist. Similarly, $\mathbb{S}_{n-1} \circ \mathbb{S}_{m+1}$ acts by filling the gap labeled $m + 1$, then filling the gap *originally* labeled $n$, if these gaps exist. These actions are the same except for the order in which the two new beads are created, which causes the two answers to differ by a sign change. As a special case, when $n = m+1$, both sides of (7) output zero because the second operator on each side tries to fill a gap that no longer exists. This completes the proof of (7). Finally, the computations in Example 1 are now justified by combining formulas (1), (3), and (7).

By applying the results in Section 3.4, we also obtain a dual theorem characterizing the action of $\mathscr{C}_\omega(\mathbb{S}_m)$ as a "gap-creation operator" or a "bead-destruction operator." Specifically, given a bead with $b$ beads strictly to its right and $g$ gaps to its left, let this bead have label $b-g$ and sign $(-1)^g$. (Note that this labeling of *beads* is related to, but different from, our prior labeling scheme for *gaps*.) If the abacus for $\mu$ has a bead labeled $m$, then $\mathscr{C}_\omega(\mathbb{S}_m)(s_\mu)$ is the sign of this bead times $s_\nu$, where we get the abacus for $\nu$ by replacing the bead labeled $m$ by a gap. If the abacus for $\mu$ has no bead labeled $m$, then $\mathscr{C}_\omega(\mathbb{S}_m)(s_\mu)$ is zero.

### 3.6 Abacus Versions of $\mathbb{H}_m$ and $\mathbb{C}_m$

With Theorem 2 in hand, we can describe how to compute $\mathbb{H}_m(s_\mu)$ and $\mathbb{C}_m(s_\mu)$ using abacus-histories. Our implementation of $\mathbb{H}_m$ is based on formula (18). Starting with the abacus for $\mu$, do the following steps in all possible ways. Choose an integer $c \geq 0$. In time step 1, apply $h_c^\perp$ by moving some beads a total of $c$ steps west (avoiding original bead positions), then moving all beads one step south. In time step 2, apply $\mathbb{S}_{m+c}$ by creating a new bead in the gap now labeled $m+c$ (if any). The Schur function corresponding to the new abacus is weighted by $q^c$ times the sign of the gap where the new bead was created.

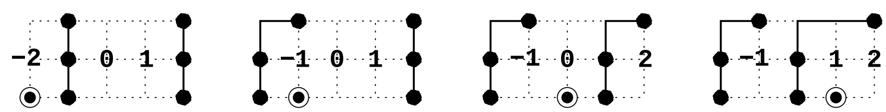


Figure 9: Computing $\mathbb{H}_{-2}(s_{(31)})$ using abacus-histories.

For example, by drawing the objects in Figure 9, we find

$$\mathbb{H}_{-2}(s_{(31)}) = +s_{(200)} - q^1 s_{(200)} - q^2 s_{(110)} + q^3 s_{(110)}. \tag{25}$$

In these abacus-histories, we have included gap labels in the middle row and circled the new beads created by $\mathbb{S}_{m+c}$. By making similar pictures, one can check that

$$\mathbb{H}_1(s_{(31)}) = -s_{(221)} + q^1 s_{(221)} + q^2 s_{(320)} + q^2 s_{(311)} + q^3 s_{(410)}. \tag{26}$$

Note that these answers are neither Schur-positive nor Schur-negative. We observe that every original bead takes two consecutive south steps in these abacus-histories. By combining these steps into a single south step, we can shorten the time needed for the $\mathbb{H}_m$ operator from two time steps to one time step. We do this from now on.

We can compute $\mathbb{C}_m(s_\mu)$ by generating exactly the same collection of abacus-histories used for $\mathbb{H}_m(s_\mu)$. The only difference is that each weight $q^c$ is replaced by $q^{-c}$, and the final answer is multiplied by the global factor $(-1/q)^{m-1}$. For example,

$$\begin{aligned}\mathbb{C}_{-2}(s_{(31)}) &= -q^3 s_{(200)} + q^2 s_{(200)} + q^1 s_{(110)} - q^0 s_{(110)};\\ \mathbb{C}_1(s_{(31)}) &= -s_{(221)} + q^{-1} s_{(221)} + q^{-2} s_{(320)} + q^{-2} s_{(311)} + q^{-3} s_{(410)}.\end{aligned}$$

### 3.7 Abacus Version of $\mathbb{B}_m$

Finally, we describe how to compute $\mathbb{B}_m(s_\mu)$ using abacus-histories. Since $\mathbb{B}_m = \mathscr{C}_\omega(\mathbb{H}_m) = \omega \circ \mathbb{H}_m \circ \omega$, one approach is to calculate $\mathbb{H}_m(s_{\mu'})$ as described earlier, then conjugate all partitions in the resulting Schur expansion. For example, using (25) and (26), we compute:

$$\begin{aligned} \mathbb{B}_{-2}(s_{(211)}) &= +s_{(11)} - q^1 s_{(11)} - q^2 s_{(2)} + q^3 s_{(2)}; \\ \mathbb{B}_1(s_{(211)}) &= -s_{(32)} + q^1 s_{(32)} + q^2 s_{(221)} + q^2 s_{(311)} + q^3 s_{(2111)}. \end{aligned}$$

Alternatively, we can use the formula

$$\mathbb{B}_m = \sum_{d \geq 0} q^d \mathscr{C}_\omega(\mathbb{S}_{m+d}) \circ e_d^\perp$$

proved in Section 2.5. Starting with the doubly-infinite abacus for $\mu$, do the following steps in all possible ways. Choose an integer $d \geq 0$. In time step 1, apply $e_d^\perp$ by moving $d$ distinct beads one step southwest while the remaining beads move one step south with no collisions. In time step 2, apply $\mathscr{C}_\omega(\mathbb{S}_{m+d})$ by destroying the bead with label $m + d$ (if any). The Schur function corresponding to the new abacus is weighted by $q^d$ times the sign of the destructed bead.

# 4 Abacus-History Models for $H_\alpha$, $C_\alpha$, $B_\alpha$, etc.

### 4.1 Combinatorial Formulas

Now that we have abacus-history interpretations for the operators $\mathbb{S}_m$, $\mathbb{H}_m$, $\mathbb{C}_m$, $\mathbb{B}_m$, $h_m^\perp$, etc., we can build abacus-history models giving the Schur expansion when any finite sequence of these operators is applied to any Schur function. We simply concatenate the diagrams for the individual operators in all possible ways and sum the signed, weighted Schur functions corresponding to the final abaci. We illustrate this process here by describing combinatorial formulas for $H_\alpha$, $C_\alpha$, $B_\alpha$, and the analogous symmetric functions $\mathbb{H}_\alpha(s_\mu)$, $\mathbb{C}_\alpha(s_\mu)$, and $\mathbb{B}_\alpha(s_\mu)$ obtained by replacing 1 by $s_\mu$ in (4), (5), and (6).

Fix a sequence of integers $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_L)$. We compute $H_\alpha = \mathbb{H}_{\alpha_1} \circ \cdots \circ \mathbb{H}_{\alpha_L}(1)$ using abacus-histories that take $L$ time steps. We start with an empty abacus (corresponding to the input $s_{(0)} = 1$) where the gap in each position $i \geq 0$ has label $i$. In time step 1, we cannot move any beads west, so we create a new bead in the gap labeled $\alpha_L$. In time step 2, we choose $c_2 \geq 0$, move the lone bead west $c_2$ steps and south once, then create a new bead in the gap now labeled $\alpha_{L-1} + c_2$. In time step 3, we choose $c_3 \geq 0$, move the two beads west a total of $c_3$ steps and south, then create a new bead in the gap now labeled $\alpha_{L-2} + c_3$. And so on. If at any stage there is no gap with the required label, then that particular diagram disappears and contributes zero to the answer. If the diagram survives through all $L$ time steps, then its final abacus contributes a Schur function weighted by $q^{c_2+c_3+\cdots+c_L}$ times the signs arising from all the bead creation steps. Thus, the final power of $q$ is the total number of west steps taken by all the beads, while the final sign is $-1$ raised to the total number of beads to the right of newly created beads in all time steps. The computation for $\mathbb{H}_\alpha(s_\mu)$ is the same, except now we start with the abacus for $\mu$ instead of an empty abacus. Here we might move some beads $c_1 \geq 0$ steps west in the first time interval, leading to the creation of a new bead in the gap now labeled $\alpha_L + c_1$. When $\mu = 0$ we must have $c_1 = 0$.

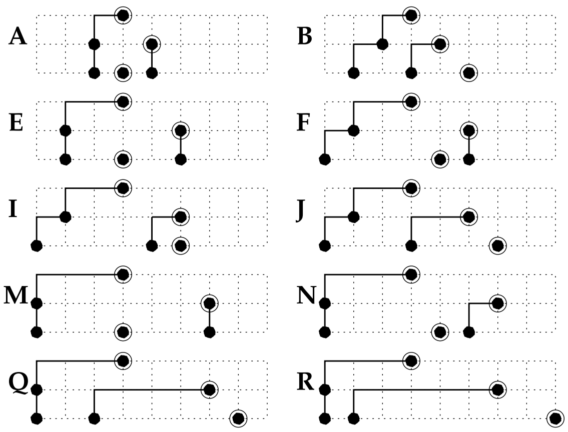


Figure 10: Computing $H_{(123)}$ using abacus-histories.

The following remarks can aid in generating the diagrams for $H_\alpha$. The *default starting positions* for the new beads are $\alpha_L, \alpha_{L-1}+1, \alpha_{L-2}+2, \ldots, \alpha_1+L-1$. These are the positions (not gap labels) on the initial abacus where new beads would appear if all $c_i$ were zero. (This follows since gap labels coincide with position numbers on an empty abacus, and each bead creation decrements all current gap labels.) The *actual starting positions* for the new beads are

$$\alpha_L + c_1, \alpha_{L-1}+1+c_2, \alpha_{L-2}+2+c_3, \ldots, \alpha_1+L+c_L;$$

these are obtained by moving each default starting position east by the number of west steps in the preceding row.

**Example 3.** Figure 10 uses abacus-histories to compute $H_{(123)} = \mathbb{H}_1(\mathbb{H}_2(\mathbb{H}_3(1)))$. For brevity, we omit the top rows, which have no beads in any positive position. All three new beads have default starting position 3. There are 18 objects in all, but we find two pairs of objects that cancel (C with F, and I with N). We are left with

$$\begin{aligned} H_{(123)} =& q^8 s_{(600)} + (q^6+q^7)s_{(510)} + (q^6+q^5+q^4-q^3)s_{(420)} \\ &+ q^5 s_{(411)} + q^5 s_{(330)} + (q^4+q^3-q^2)s_{(321)} + (q^2-q)s_{(222)}. \end{aligned}$$

We compute $C_\alpha$ (resp. $\mathbb{C}_\alpha(s_\mu)$) by making exactly the same abacus-histories used for $H_\alpha$ (resp. $\mathbb{H}_\alpha(s_\mu)$). The only difference is that the $q$-weight of each abacus-history is now $q^{-(c_1+\cdots+c_L)}$ and the final answer must be multiplied by $(-1/q)^{|\alpha|-\ell(\alpha)}$, where $|\alpha| = \alpha_1+\cdots+\alpha_L$ and $\ell(\alpha) = L$.

Since $\mathbb{B}_m = \mathscr{C}_\omega(\mathbb{H}_m)$ for every integer $m$, we can compute $\mathbb{B}_{\alpha_1} \circ \cdots \circ \mathbb{B}_{\alpha_L}(s_\mu)$ by applying $\mathbb{H}_\alpha$ to $s_{\mu'}$ as described above, then conjugating all partitions in the resulting Schur expansion. Alternatively, we can chain together the moves for the $\mathbb{B}_{\alpha_i}$ described at the end of Section 3.7. Beware that $B_\alpha$ (as defined in [4]) is found by starting with 1 and applying $\mathbb{B}_{\alpha_1}, \mathbb{B}_{\alpha_2}, \ldots, \mathbb{B}_{\alpha_L}$ in this order.

## 4.2 Application to Three-Row Hall–Littlewood Polynomials

As we have seen, for general $\alpha$ the Schur expansion of $H_\alpha$ has a mixture of positive and negative terms. But for partitions $\nu$, a celebrated theorem of Lascoux and Schützenberger [10] shows

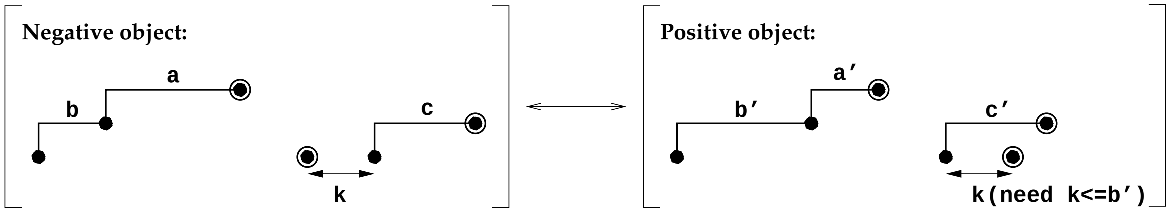


Figure 11: An involution on abacus-histories with three beads.

that all Schur coefficients of $H_\nu$ are polynomials in $q$ with nonnegative integer coefficients (the *Kostka–Foulkes polynomials*). According to this theorem, the coefficient of $s_\lambda$ in $H_\nu$ is the sum of $q^{\mathrm{charge}(T)}$ over all semistandard Young tableaux $T$ of shape $\lambda$ and content $\nu$, where charge is computed from $T$ by an explicit combinatorial rule (see [1], [15, pg. 242], and [16, Sec. 1.7] for more details). Kirillov and Reshetikhin [8, 9] gave another combinatorial formula for the Kostka–Foulkes polynomials as a sum over *rigged configurations* weighted by a suitable charge statistic. A detailed survey of combinatorial formulas for Hall–Littlewood polynomials and their applications to representation theory may be found in [2].

Our combinatorial formula for $H_\alpha$ based on abacus-histories holds for general integer sequences $\alpha$, but it is not manifestly Schur-positive when $\alpha$ happens to be an integer partition. On the other hand, our $q$-statistic (the total number of west steps in the object) is much simpler to work with compared to the complicated charge statistic on tableaux. As an application of our abacus-history model, we now give a simple bijective proof of the Schur-positivity of $H_\nu$ when $\nu$ is a partition with at most three parts.

If $\nu$ has only one part, then it is immediate that $H_\nu = \mathbb{H}_{\nu_1}(1) = s_{\nu_1}$. Next suppose $\nu = (\nu_1 \geq \nu_2)$ has two parts. When computing $H_\nu$ via abacus-histories, the default starting positions of the two beads are $\nu_2$ and $\nu_1 + 1 > \nu_2$. When the second bead is created, the first bead has moved to a column $\nu_2 - c_2$ for some $c_2 \geq 0$, and the second bead actually starts in column $\nu_1 + 1 + c_2 > \nu_2 - c_2$. This means that all abacus-histories for $H_\nu$ have positive sign, and our formula is already Schur-positive in this case.

Now let $\nu = (\nu_1 \geq \nu_2 \geq \nu_3)$ have three parts. As before, since $\nu_2 \geq \nu_3$, the second bead always gets created to the right of the first bead's current column. For similar reasons, the bead created third must start to the right of the first bead in the lowest row. But it is possible that this third bead appears to the left of the second bead's position in that row, leading to a negative object that must be canceled.

We cancel these objects using the involution suggested in Figure 11. Given a negative object as just described, let $k$ be the distance between the new bead in the lowest row and the bead to its right. Let $a, b, c \geq 0$ count west steps as shown on the left in the figure, so bead 1's path is $\mathrm{W}^a\mathrm{S}\mathrm{W}^b\mathrm{S}$ and bead 2's path is $\mathrm{W}^c\mathrm{S}$. The involution acts by replacing $a$ by $a-k$ and $b$ by $b+k$, which causes bead 2's actual starting position to move left $k$ columns and bead 3's actual starting position to move right $k$ columns. This action matches the given negative object with a positive object having the same number of west steps (hence the same $q$-power) and the same bead positions on the final abacus (hence the same Schur function).

Going the other way, consider a positive object (as shown on the right in Figure 11) where the new bead in the bottom row is $k$ columns to the right of the second bead, bead 1's path is $\mathrm{W}^{a'}\mathrm{S}\mathrm{W}^{b'}\mathrm{S}$, and bead 2's path is $\mathrm{W}^{c'}\mathrm{S}$. If $k \leq b'$, then the involution acts by replacing $a'$ by

$a' + k$ and $b'$ by $b' - k$, causing the other two paths to switch places as before. If $k > b'$, then this positive object is a fixed point of the involution.

To see that this involution really works, we must check a few items. Fix an arbitrary negative object with notation as in Figure 11. First, we must show $k \leq a$, since $a'$ is not allowed to be negative. On one hand, the actual starting position of the new bead in the bottom row is $\nu_1 + 2 + b + c$. On the other hand, the bead created in the middle row starts in position $\nu_2 + 1 + a$ and ends in the bottom row in position $\nu_2 + 1 + a - c$. Therefore,

$$k = (\nu_2 + 1 + a - c) - (\nu_1 + 2 + b + c) = a - (b + 2c + 1 + \nu_1 - \nu_2). \tag{27}$$

Since $b, c \geq 0$ and $\nu_1 \geq \nu_2$, the quantity subtracted from $a$ is strictly positive, so we actually have $k < a$.

Second, we show that applying the involution to a negative object does not cause the first two beads to collide in the middle row. After replacing $a$ by $a' = a - k$, the first bead moves from the top row to the middle row in position $\nu_3 - a' = \nu_3 - a + k$. The bead created in the middle row now starts in position $\nu_2 + 1 + a' = \nu_2 + 1 + a - k$ and moves left $c' = c$ steps to position $\nu_2 + 1 + a - k - c$ before moving down to the bottom row. Therefore, to avoid a bead collision, we need $\nu_3 - a + k < \nu_2 + 1 + a - k - c$, or equivalently $2(a - k) > c + \nu_3 - \nu_2 - 1$. Using (27) to substitute for $a - k$, we need $2(b + 2c + 1 + \nu_1 - \nu_2) > c + \nu_3 - \nu_2 - 1$, which rearranges to $2b + 3c + 3 + 2\nu_1 - \nu_2 - \nu_3 > 0$. This is true, since $b, c \geq 0$ and $\nu_1 \geq \nu_2 \geq \nu_3$.

Third, we show that applying the involution twice restores the original object. This follows since the value of $k$ is the same for the object and its image, and $b' = b + k$ automatically satisfies $k \leq b'$. We have now proved that our involution is well-defined and cancels all negative objects.

By analyzing the fixed points of this involution more closely, we can prove the following explicit formula for the Schur coefficients of $H_\nu$ when $\nu$ is a three-part partition.

**Theorem 4.** *For all partitions $\nu = (\nu_1 \geq \nu_2 \geq \nu_3 > 0)$ and $\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0)$ such that $|\lambda| = |\nu|$, the coefficient of $s_\lambda$ in $H_\nu$ is*

$$\sum_{b=0}^{\min(\lambda_1-\lambda_2,\lambda_2-\lambda_3,\nu_3-\lambda_3,\lambda_1-\nu_1)} q^{\nu_3-\lambda_3+\lambda_1-\nu_1-b}. \tag{28}$$

*Proof.* We fix $\lambda, \nu$ as in the theorem statement and enumerate the positive fixed points of the involution. To obtain an uncanceled term $s_\lambda$ in the computation of $H_\nu$, the beads in the abacus-history must move as follows. The first bead starts in position $\nu_3$ and ends in position $\lambda_3$ after moving along some path $\mathrm{W}^a\mathrm{SW}^b\mathrm{S}$. (We switch here to unprimed parameters for the positive fixed point.) The second bead starts in position $\nu_2 + 1 + a$ and ends in position $\lambda_2 + 1$ (since all negative objects cancel) after moving along some path $\mathrm{W}^c\mathrm{S}$. The third bead starts in position $\nu_1 + 2 + b + c$ and ends (without moving) in position $\lambda_1 + 2$. We deduce that $\lambda_3 = \nu_3 - a - b$, $\lambda_2 + 1 = \nu_2 + 1 + a - c$, and $\lambda_1 + 2 = \nu_1 + 2 + b + c$. Since $\lambda$ and $\nu$ are fixed, the entire object is uniquely determined once we select the value of $b$. Eliminating $a$ and $c$, we see that the $q$-weight of the object is

$$q^{a+b+c} = q^{\nu_3-\lambda_3+\lambda_1-\nu_1-b}.$$

Which choices of $b$ are allowed? We certainly need $b \geq 0$, as well as $a \geq 0$ and $c \geq 0$. Using $a = \nu_3 - \lambda_3 - b$ and $c = \lambda_1 - \nu_1 - b$, the conditions on $a$ and $c$ are equivalent to $b \leq \nu_3 - \lambda_3$ and $b \leq \lambda_1 - \nu_1$. We also need the first two beads not to collide in the middle row, so we need $\nu_3 - a < \nu_2 + 1 + a - c$. This condition is equivalent to $b + \lambda_3 < \lambda_2 + 1$ and, hence, to $b \leq \lambda_2 - \lambda_3$.

Finally, letting $k = (\lambda_1 + 2) - (\lambda_2 + 1)$ be the distance between the two rightmost beads in the bottom row, we need $k > b$ for this positive object to be a fixed point of the involution. This condition rearranges to $b \leq \lambda_1 - \lambda_2$. Combining the five conditions on $b$ leads to the summation in the theorem statement. □

It is also possible to derive (28) starting from the charge formula for the Schur expansion of $H_\nu$. But such a proof is quite tedious, requiring a messy case analysis due to the complicated definition of the charge statistic.

One might ask if the involution for three-row shapes extends to partitions $\nu$ with more than three parts. While the same involution certainly applies to the first three rows of larger abacus-histories, more moves are needed to eliminate all negative objects. Even in the case of four-row shapes, the new cancellation moves are much more intricate than the move described here. We leave it as an open question to reprove the Schur-positivity of $H_\nu$ for all partitions $\nu$ via an explicit involution on abacus-histories. It would also be interesting to find a specific weight-preserving bijection between the set of fixed points for such an involution and the set of semistandard tableaux.

## 5 Appendix: Proofs of Two Plethystic Formulas

This appendix proves the equivalence of the plethystic definitions and the algebraic definitions of $\mathbb{S}_m$ and $\mathbb{C}_m$. We require just three basic plethystic identities (see [14] for a detailed exposition of plethystic notation including proofs of these facts). First, for any alphabets $A$ and $B$ and any partition $\mu$, we have the plethystic addition rule

$$s_\mu[A+B] = \sum_{\nu\colon \nu\subseteq\mu} s_\nu[A] s_{\mu/\nu}[B].$$

Second, for formal variables $q$ and $z$ and partitions $\nu \subseteq \mu$,

$$s_{\mu/\nu}[q/z] = \begin{cases} (q/z)^{|\mu/\nu|} & \text{if } \mu/\nu \text{ is a horizontal strip;} \\ 0 & \text{otherwise.} \end{cases}$$

Third, for all $\nu \subseteq \mu$,

$$s_{\mu/\nu}[-1/z] = (-1)^{|\mu/\nu|} s_{\mu'/\nu'}[1/z] = \begin{cases} (-1/z)^{|\mu/\nu|} & \text{if } \mu/\nu \text{ is a vertical strip;} \\ 0 & \text{otherwise.} \end{cases}$$

Recall that HS($c$) (resp. VS($c$)) is the set of horizontal (resp. vertical) strips with $c$ cells.

We prove the equivalence of the definitions (15) and (16) for $\mathbb{S}_m$ by showing that the two formulas have the same action on the Schur basis. Taking $P = s_\mu$ in (15) and using the rules above, we compute:

$$\begin{aligned}\mathbb{S}_m(s_\mu) &= s_\mu[X-(1/z)]\sum_{k\geq 0} h_k z^k \Big|_{z^m} = \sum_{\nu\subseteq\mu} s_\nu[X] s_{\mu/\nu}[-1/z]\sum_{k\geq 0} h_k z^k \Big|_{z^m} \\ &= \sum_{c\geq 0}\sum_{\substack{\nu\subseteq\mu:\\ \mu/\nu\in\mathrm{VS}(c)}} (-1/z)^c \sum_{k\geq 0} s_\nu h_k z^k \Big|_{z^m} = \sum_{c\geq 0}\sum_{\substack{\nu\subseteq\mu:\\ \mu/\nu\in\mathrm{VS}(c)}} (-1)^c s_\nu h_{m+c} \\ &= \sum_{c\geq 0} (-1)^c M_{h_{m+c}}(e_c^\perp(s_\mu)).\end{aligned}$$

This agrees with (16).

Now we prove the equivalence of (17) and (18). Applying (17) to $P = s_\mu$ gives:

$$\begin{aligned}\mathbb{H}_m(s_\mu) &= s_\mu[(X-1/z)+q/z]\sum_{k\geq 0} h_k z^k \Big|_{z^m} = \sum_{\nu\subseteq\mu} s_\nu[X-1/z] s_{\mu/\nu}[q/z]\sum_{k\geq 0} h_k z^k \Big|_{z^m} \\ &= \sum_{c\geq 0}\sum_{\substack{\nu\subseteq\mu:\\ \mu/\nu\in\mathrm{HS}(c)}} (q/z)^c s_\nu[X-1/z]\sum_{k\geq 0} h_k z^k \Big|_{z^m} \\ &= \sum_{c\geq 0} q^c \sum_{\substack{\nu\subseteq\mu:\\ \mu/\nu\in\mathrm{HS}(c)}} s_\nu[X-1/z]\sum_{k\geq 0} h_k z^k \Big|_{z^{m+c}} = \sum_{c\geq 0} q^c \sum_{\substack{\nu\subseteq\mu:\\ \mu/\nu\in\mathrm{HS}(c)}} \mathbb{S}_{m+c}(s_\nu) \\ &= \sum_{c\geq 0} q^c\, \mathbb{S}_{m+c}\left(\sum_{\nu\subseteq\mu:\ \mu/\nu\in\mathrm{HS}(c)} s_\nu\right) = \sum_{c\geq 0} q^c\, \mathbb{S}_{m+c}(h_c^\perp(s_\mu)).\end{aligned}$$

This agrees with (18).

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