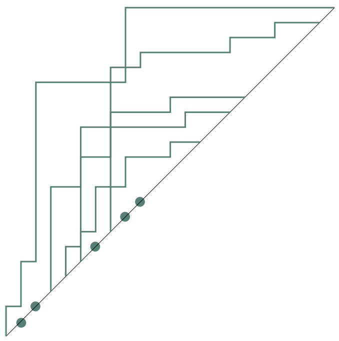
Combinatorial structures associated to the nabla operator



BIRS

September 10, 2007

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Joint with: **Nick Loehr**, Virginia Tech

Verbal stylings by: **Jim Haglund**, University of Pennsylvania

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[a] Responsible party for any errors

Representation Theory

Symmetric Functions

Combinatorics

$$V(\mu)$$

Garsia-Haiman modules

Δ_μ

Let $\{(r_i, c_i)\}_{1 \le i \le n}$ be the coordinates of the boxes of a Ferrers diagram μ.

Set

$$\Delta_\mu = \left| x_i^{r_j} y_i^{c_j} \right|_{1 \le i,j \le n}.$$

Let $V(\mu)$ be the linear span of Δ_μ along with its partial derivatives of all orders.

$V(\mu)$ example

$\boldsymbol{\mu} =$

(1,0)	
(0,0)	(0,1)

$$\boldsymbol{\Delta_\mu} = \begin{vmatrix} 1 & y_1 & x_1 \\ 1 & y_2 & x_2 \\ 1 & y_3 & x_3 \end{vmatrix}$$

$$\boldsymbol{\Delta_\mu} = (y_2 x_3 - x_2 y_3) - (y_1 x_3 - y_3 x_1) + (y_1 x_2 - y_2 x_1).$$

$V(\mu)$ example

$$\begin{array}{lcc} \mathbf{1} & \rightsquigarrow & \Delta_\mu \\ \partial_{x_i} & \rightsquigarrow & \{y_3 - y_2, y_3 - y_1, y_2 - y_1\} \\ \partial_{y_i} & \rightsquigarrow & \{x_3 - x_2, x_3 - x_1, x_2 - x_1\} \\ \partial_{x_i y_j} & \rightsquigarrow & \pm 1 \text{ or } 0 \text{ if } i = j \end{array}$$

The $n!$ Theorem(Haiman '01):

If $\mu \vdash n$, then $\dim(V(\mu)) = n!$.

S_n-action on $V(\mu)$

Let $R = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$.
R is an S_n-module via the "diagonal action":

$$\sigma x_i = x_{\sigma_i} \qquad \sigma y_i = y_{\sigma_i}.$$

$V(\mu)$ is an S_n-submodule bigraded by total x and y degree.

Some Series

Set $\boldsymbol{V}(\boldsymbol{\mu}) = \oplus_{i,j\geq 0} \boldsymbol{V}^{i,j}(\boldsymbol{\mu})$.

$$\mathbf{Hilb}(\boldsymbol{V}(\boldsymbol{\mu})) = \sum_{i,j\geq 0} \mathbf{dim}(\boldsymbol{V}^{i,j}(\boldsymbol{\mu}))\boldsymbol{t}^i \boldsymbol{q}^j$$

$$\mathbf{Frob}(\boldsymbol{V}(\boldsymbol{\mu})) =$$

$$\sum_{i,j\geq 0} \boldsymbol{t}^i \boldsymbol{q}^j \sum_{\lambda \vdash n} \boldsymbol{s}_{\lambda} \mathbf{Mult}[\chi^{\lambda}, \boldsymbol{V}^{i,j}(\boldsymbol{\mu})]$$

Representation Theory | **Symmetric Functions** | **Combinatorics**

$$\boldsymbol{V(\mu)} \xrightarrow{\text{Frob}} \boldsymbol{\tilde{H}_\mu}$$

$C = \tilde{H}$ Conjecture(Garsia-Haiman, '93):

$$\mathbf{Frob}(\boldsymbol{V(\mu)}) = \boldsymbol{\tilde{H}_\mu}.$$

Proved by Haiman, '01.

Representation Theory | **Symmetric Functions** | **Combinatorics**

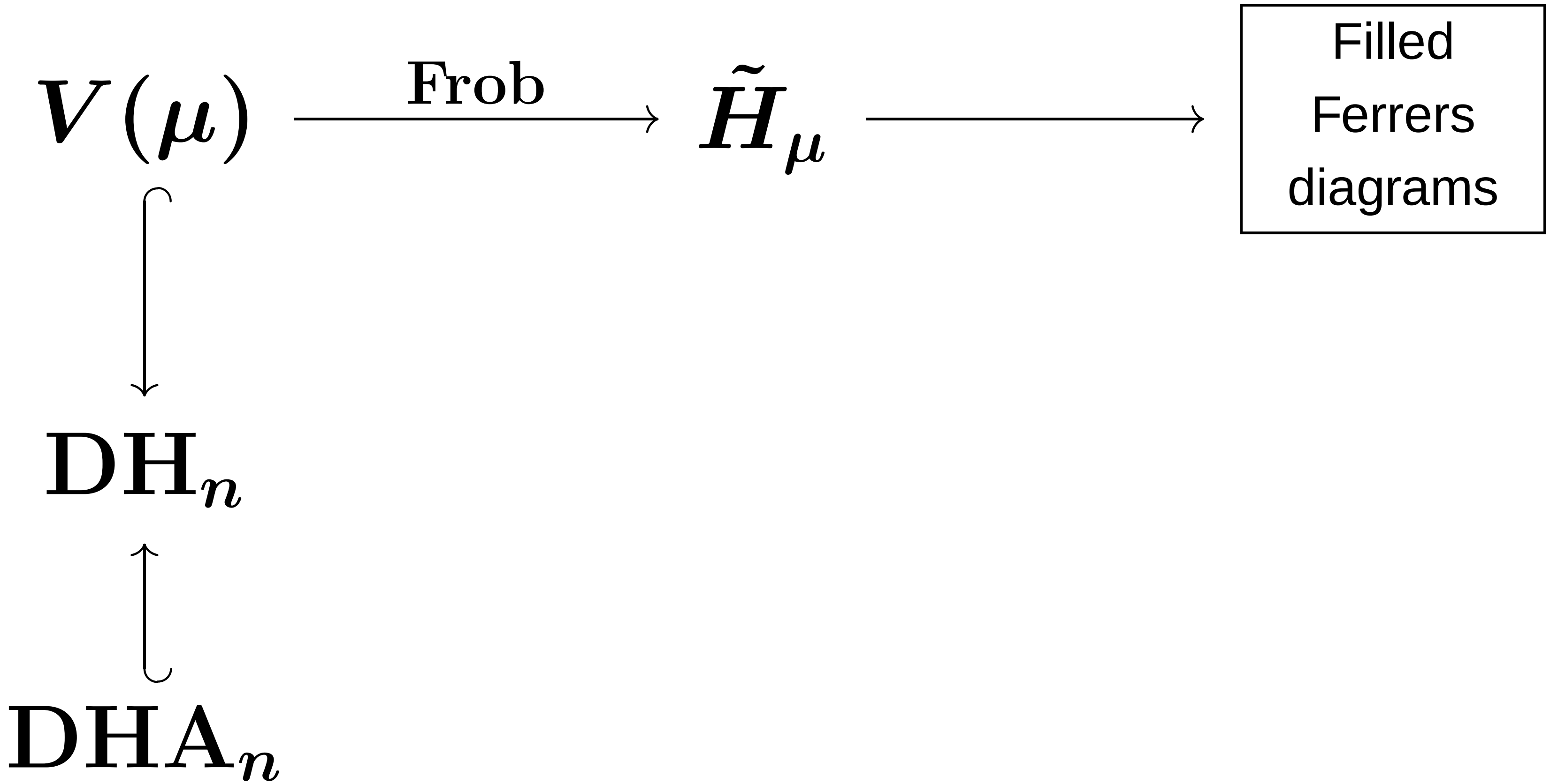
$$\boldsymbol{V(\mu)} \xrightarrow{\text{Frob}} \tilde{\boldsymbol{H}}_{\boldsymbol{\mu}} \longrightarrow \boxed{\text{Filled Ferrers diagrams}}$$

Conjecture(Haglund, ’04):

$$\tilde{\boldsymbol{H}}_{\boldsymbol{\mu}} = \sum_{\boldsymbol{T}} \boldsymbol{q}^{\mathrm{inv}_{\boldsymbol{\mu}}(\boldsymbol{T})} \boldsymbol{t}^{\mathrm{maj}_{\boldsymbol{\mu}}(\boldsymbol{T})} \boldsymbol{x}^{\boldsymbol{T}}.$$

Proved by Haglund-Haiman-Loehr, ’05.

Representation Theory | **Symmetric Functions** | **Combinatorics**



Diagonal harmonics

"Diagonal harmonics":

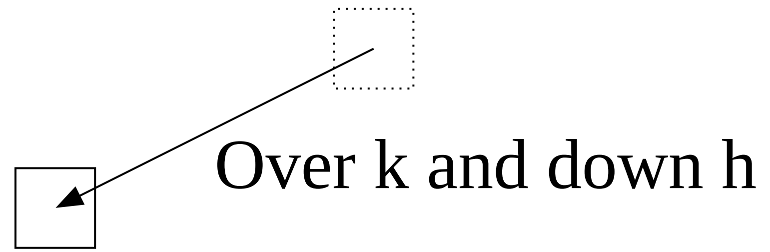
$$\mathbf{DH}_n = \left\{ f \in R : \sum_{i=1}^{n} \partial_{x_i}^{h} \partial_{y_i}^{k} f = 0, \forall h + k > 0 \right\}$$

"Diagonal harmonic alternants":

$$\mathbf{DHA}_n = \{ f \in \mathbf{DH}_n : \sigma f = \mathrm{sgn}(\sigma) f, \forall \sigma \in S_n \}$$

$V(\mu) \subset \mathbf{DH}_n$

Action of $\sum_i \partial_{x_i}^h \partial_{y_i}^k$:



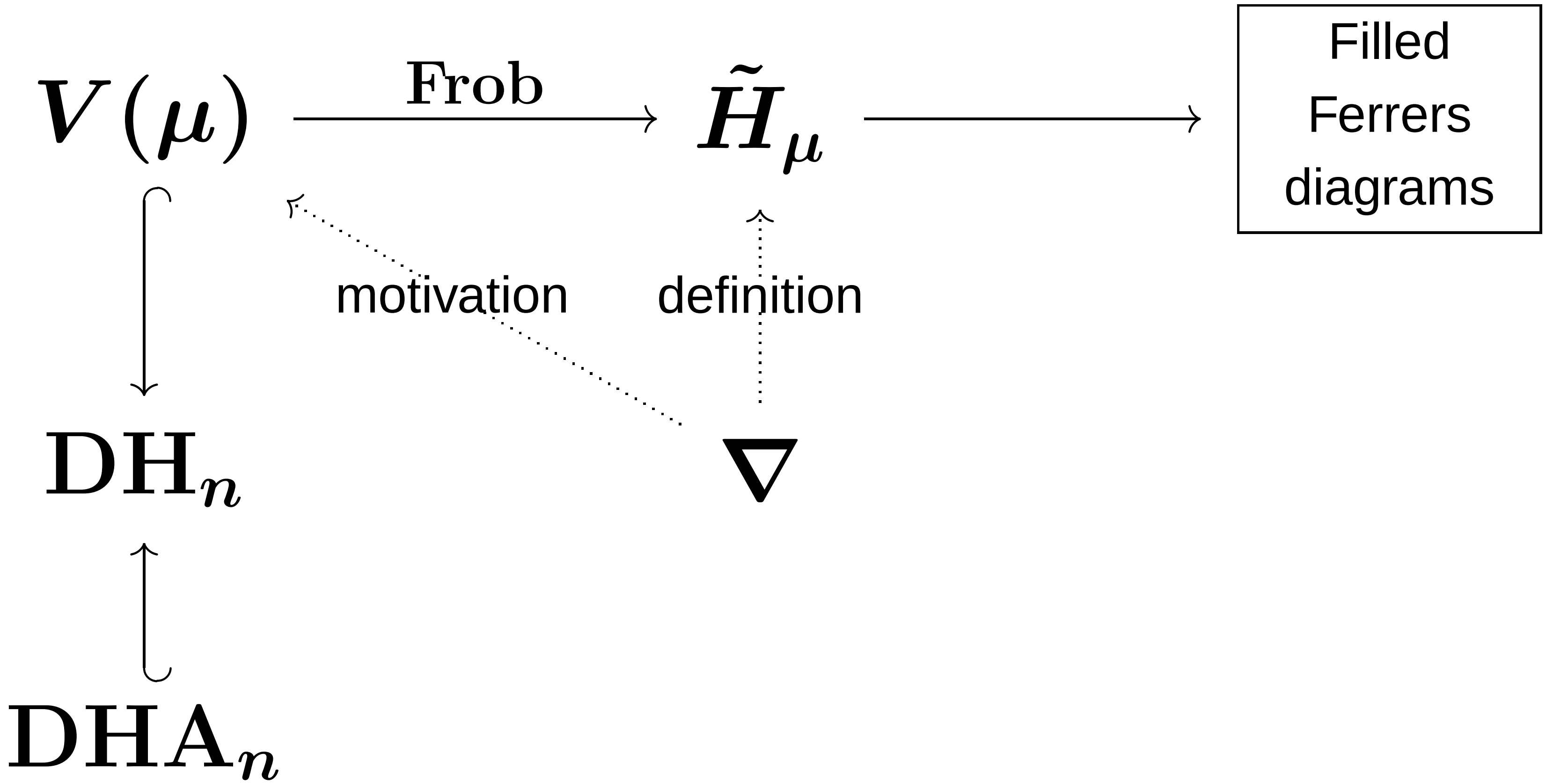
Applying to Δ_μ, we get zero if either

- a box leaves the first quadrant, or
- two boxes end up in the same place (Δ_μ is antisymmetric)

Representation Theory

Symmetric Functions

Combinatorics



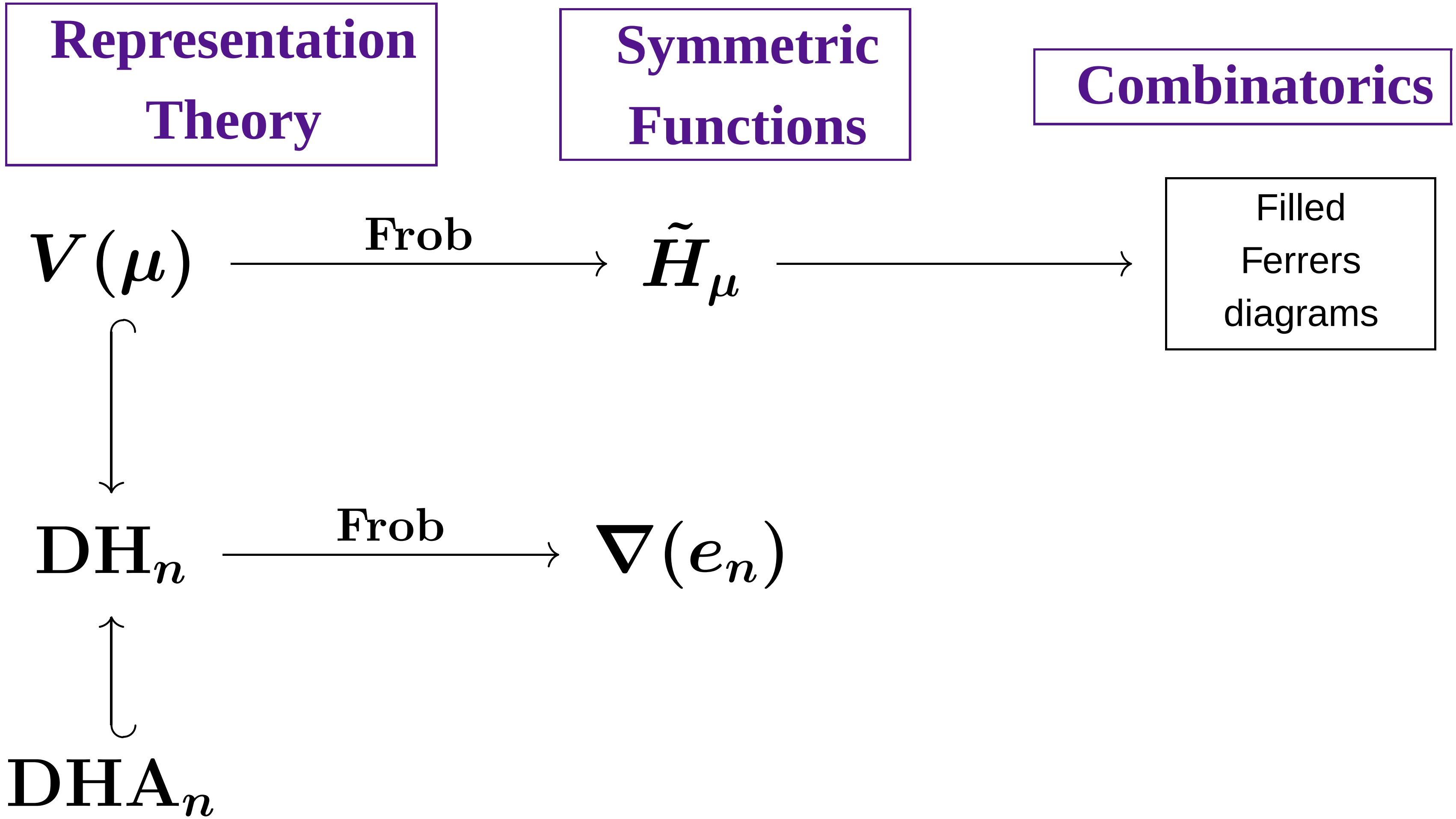
∇

Motivation:

Defined by F. Bergeron and Garsia to study $\boldsymbol{V(\mu)}$ intersections.

Definition:

$\boldsymbol{\nabla(\tilde{H}_\mu) = T_\mu \tilde{H}_\mu}$, where $\boldsymbol{T_\mu \in \mathbb{Z}[q,t]}$.



Conjecture(Garsia-Haiman, ’96):

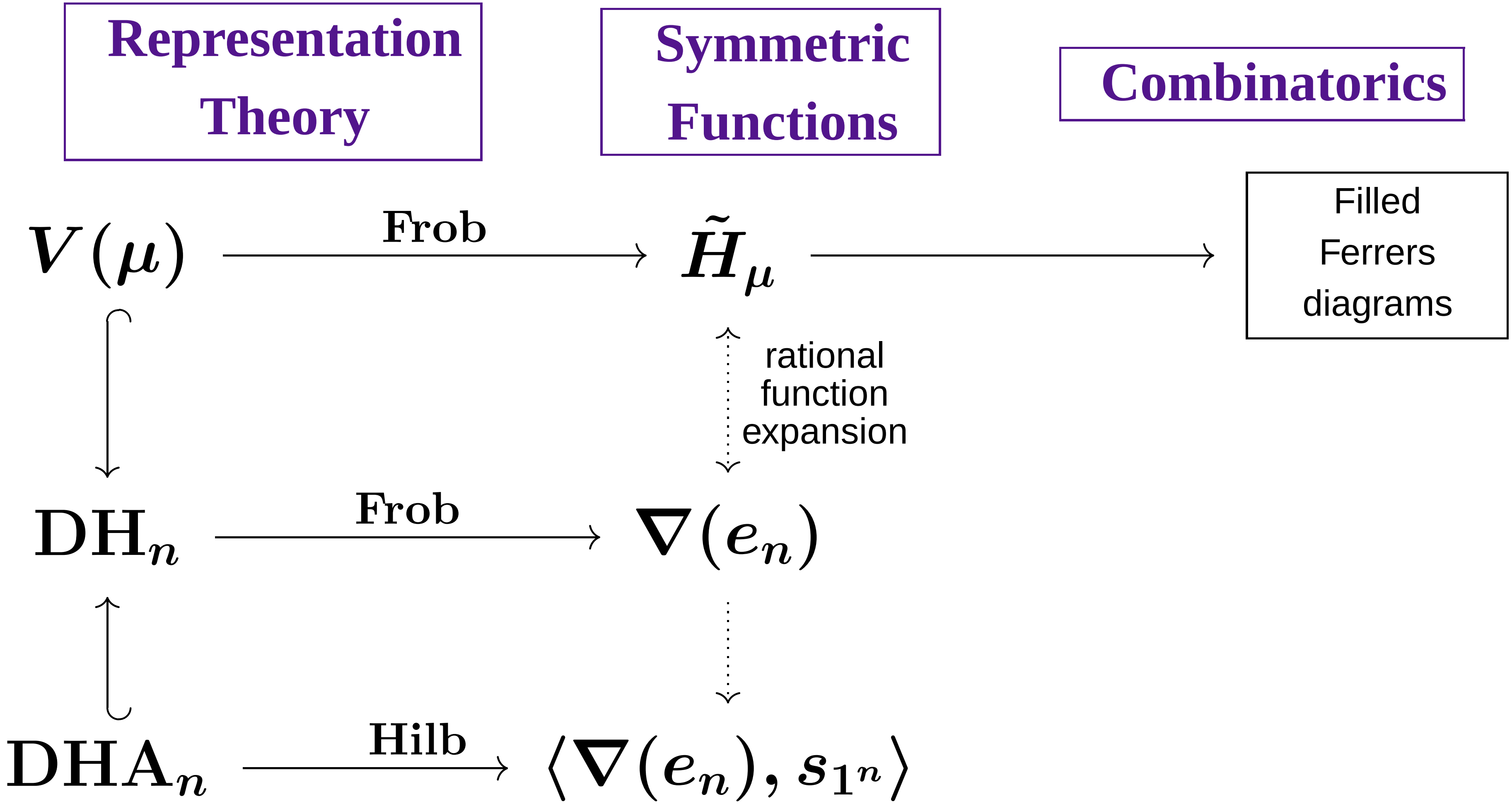
$$\mathrm{Frob}(\mathrm{DH}_n) = \nabla(e_n)$$

Proved by Haiman, ’02.

Representation Theory | **Symmetric Functions** | **Combinatorics**

$$
\begin{array}{ccccc}
V(\mu) & \xrightarrow{\textbf{Frob}} & \tilde{H}_\mu & \longrightarrow & \boxed{\text{Filled Ferrers diagrams}} \\
\big\downarrow & & & & \\
\mathbf{DH}_n & \xrightarrow{\textbf{Frob}} & \nabla(e_n) & & \\
\big\uparrow & & & & \\
\mathbf{DHA}_n & \xrightarrow{\textbf{Hilb}} & \langle \nabla(e_n), s_{1^n} \rangle & &
\end{array}
$$

The bottom row is a special case of the middle row.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$V(\mu)$
Frob
$\tilde{H}_\mu$
Filled Ferrers diagrams
rational function expansion
DH_n
Frob
$\nabla(e_n)$
DHA_n
Hilb
$\langle \nabla(e_n), s_{1^n} \rangle$

Rational Function Expansions

Theorem(Garsia-Haiman):

$$\nabla(e_n) = \sum_{\mu \vdash n} \left(\frac{T_\mu M B_\mu \Pi_\mu}{w_\mu} \right) \tilde{H}_\mu.$$

Using the fact that $\langle \tilde{H}_\mu, s_{1^n} \rangle = T_\mu$,

$$\langle \nabla(e_n), s_{1^n} \rangle = \sum_{\mu \vdash n} \frac{T_\mu^2 M B_\mu \Pi_\mu}{w_\mu} \in \mathbb{Q}(q,t)$$

This last formula defines the q,t-Catalan.

q,t-Catalan

Conjecture (Garsia-Haiman, ’92):

$$C_n(q,t) \in \mathbb{N}[q,t]$$

Proved by Garsia-Haglund, ’01.

q, t-Catalan specializations

Garsia-Haiman showed:

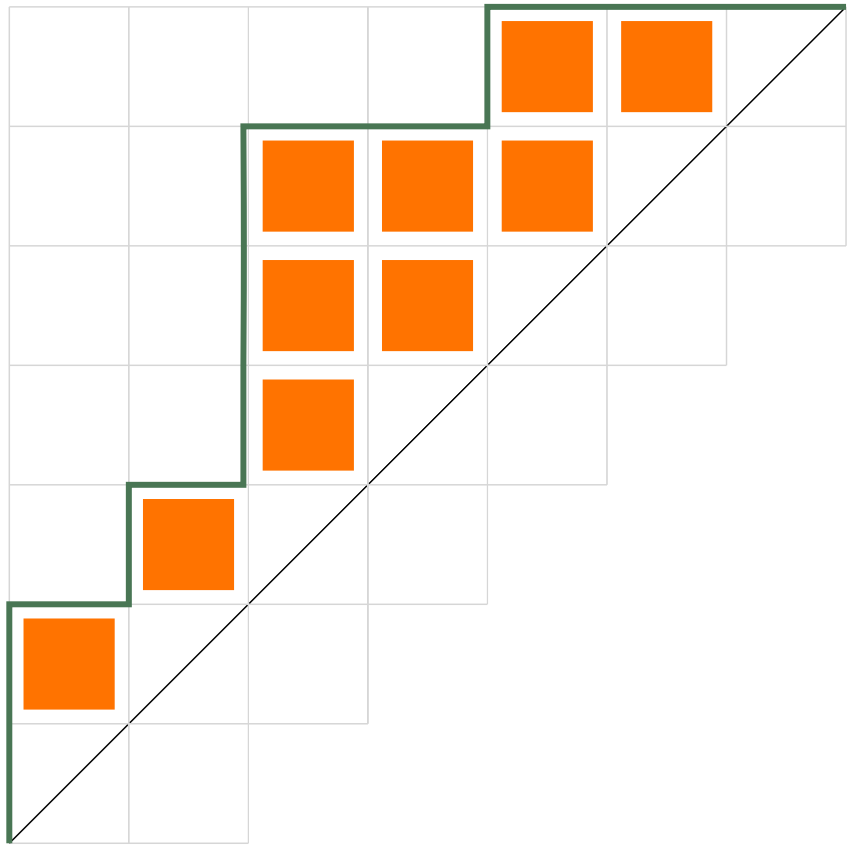
$$C_n(q,t) = C_n(t,q),$$

$$C_n(1,1) = \frac{1}{n+1}\binom{2n}{n} = C_n,$$

$$q^{\binom{n}{2}} C_n(q,1/q) = \frac{1}{[n+1]_q}\begin{bmatrix} 2n \\ n,n \end{bmatrix}_q,$$

$$C_n(1,q) = C_n(q,1) = \sum_{D \in \mathcal{D}_n} q^{\mathrm{area}(D)}.$$

Area



$$\mathbf{area} = 10$$

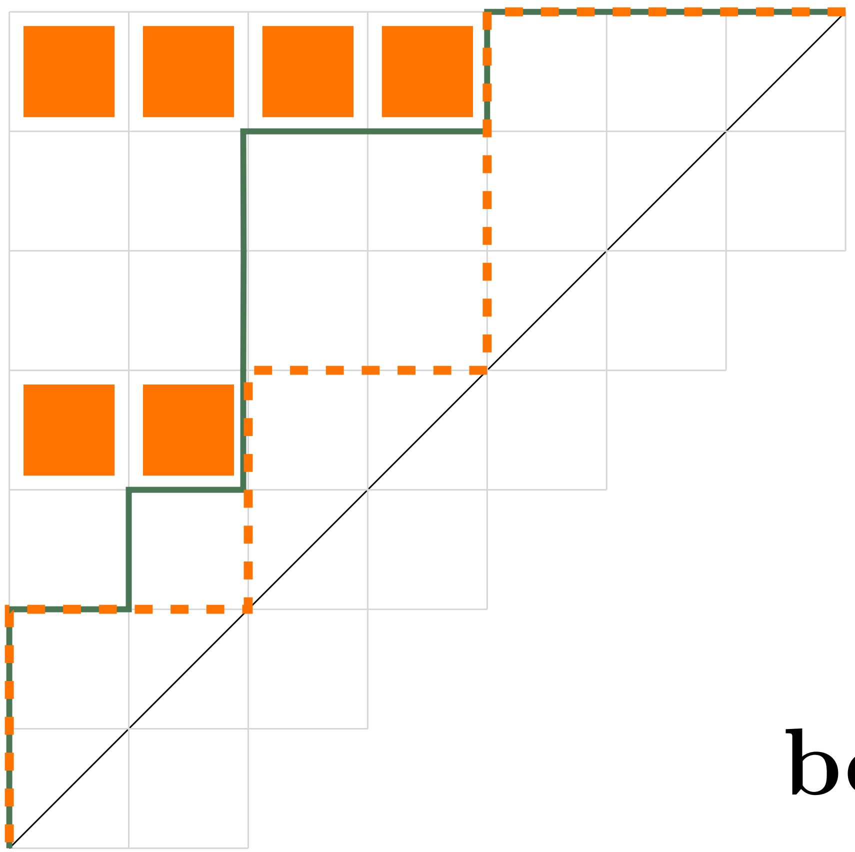
Looking for a $\mathtt{tstat}$

Wanted: A “$\mathbf{tstat}$” such that

$$C_n(q,t) = \sum_{D \in \mathcal{D}_n} q^{\mathrm{area}(D)} t^{\mathrm{tstat}(D)}.$$

Haglund proposed “bounce” for $\mathbf{tstat}$.

Bounce



$$\mathbf{boun} = \mathbf{4} + \mathbf{2} + \mathbf{0}$$

area-boun Conjecture

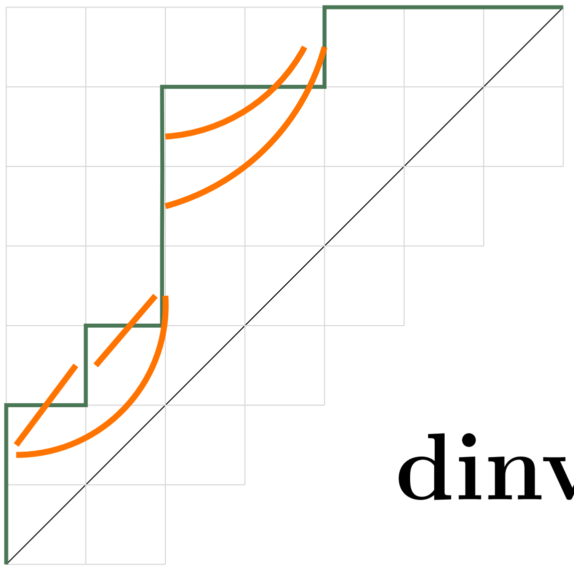
Conjecture (Haglund, ’00):

$$C_n(q,t) = \sum_{D \in \mathcal{D}_n} q^{\mathrm{area}(D)} t^{\mathrm{boun}(D)}$$

Proved by Garsia-Haglund, ’01.

Dinv

Haiman's "$\mathbf{dinv}$":

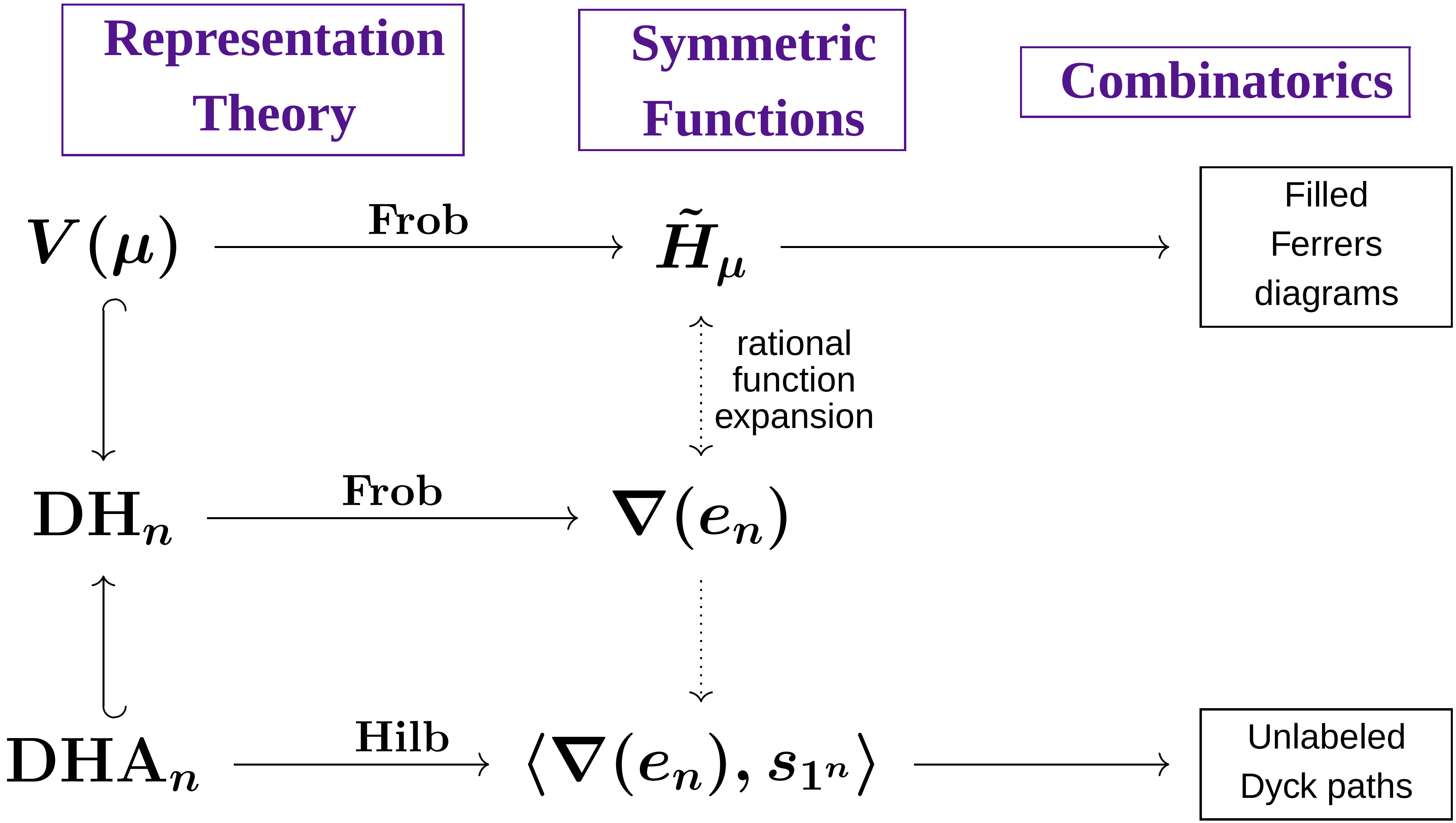


$$\mathbf{dinv} = \mathbf{5}$$

Note: There exists a bijection $\phi : \mathcal{D}_n \longrightarrow \mathcal{D}_n$ taking

$$(\mathbf{dinv}(D), \mathbf{area}(D)) \mapsto$$

$$(\mathbf{area}(\phi(D)), \mathbf{boun}(\phi(D))).$$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Unlabeled Dyck paths

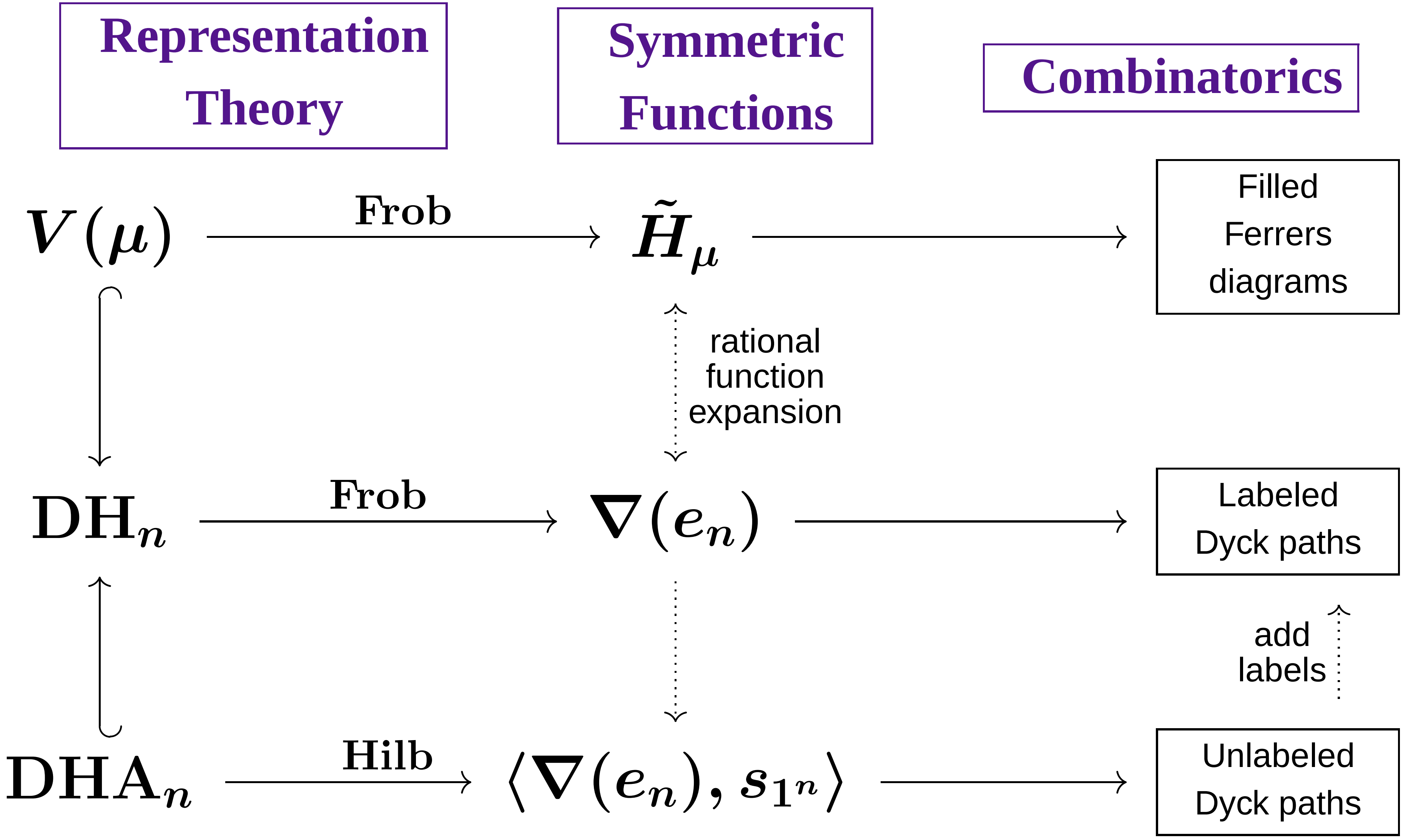
Labeling paradigm

For a symmetric function $\boldsymbol{a}$,

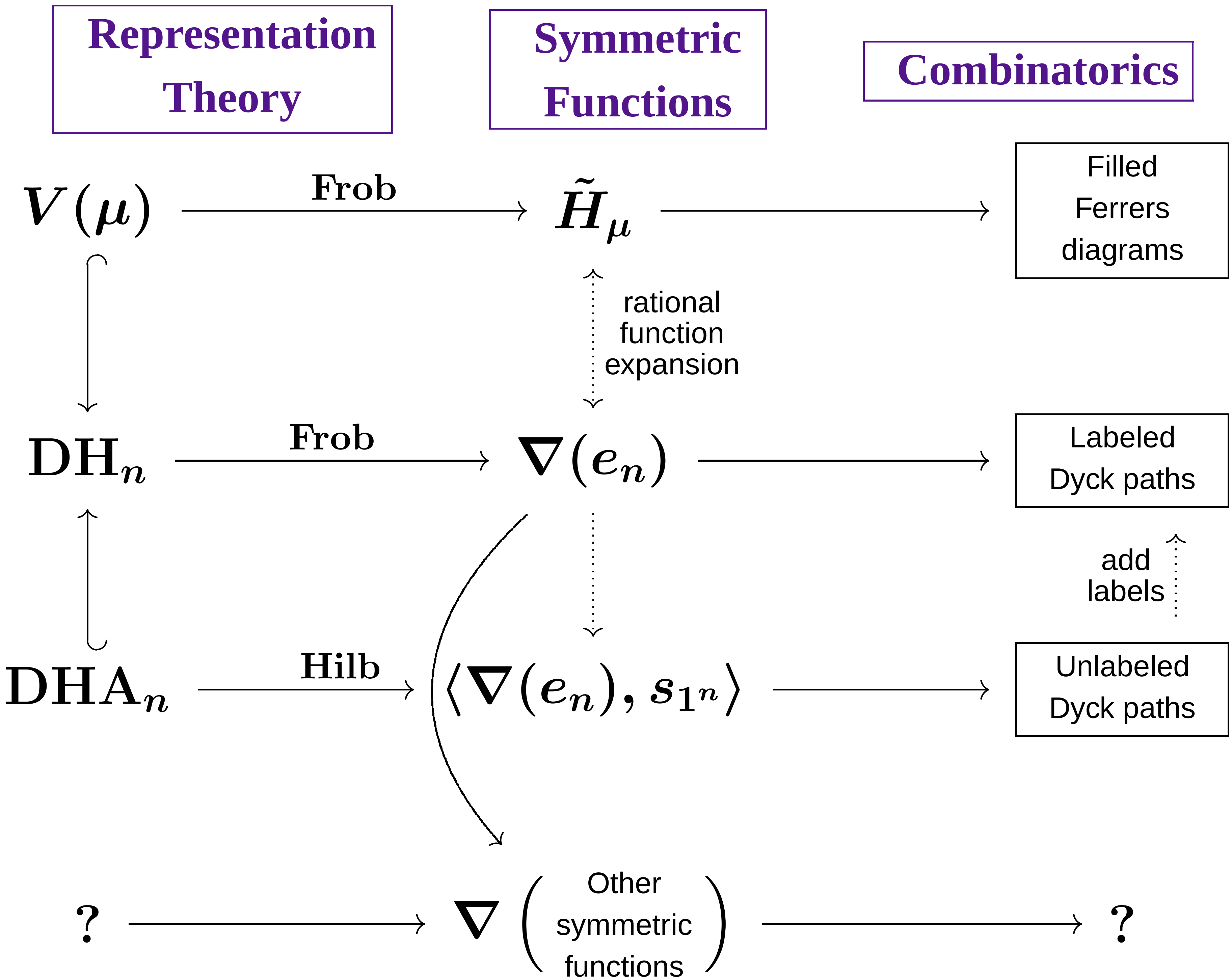
$$\langle \boldsymbol{\nabla}(\boldsymbol{a}), \boldsymbol{s}_{1^n} \rangle \longrightarrow \text{Objects, unlabeled}$$

$$\langle \boldsymbol{\nabla}(\boldsymbol{a}), \boldsymbol{h}_{1^n} \rangle \longrightarrow \text{Objects, distinct labels}$$

$$\boldsymbol{\nabla}(\boldsymbol{a}) \longrightarrow \text{Objects, repeated labels}$$



Note: $\mathbf{dinv}$ and $\mathbf{boun}$ have “labeled” versions



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Labeled Dyck paths
add labels
DHA_n
Hilb
$\langle \nabla(e_n), s_{1^n} \rangle$
Unlabeled Dyck paths
?
∇ (Other symmetric functions)
?

Conjecturally known nablas

$\nabla(e_n)$ → Labeled Dyck paths

Haglund-Haiman-Loehr-Remmel-Ulyanov

Proof for $\langle \nabla(e_n), s_{1^n} \rangle$: Garsia-Haglund-Haiman

$\nabla(p_n)$ → Labeled square paths

Loehr-W

Proof for $\langle \nabla(p_n), s_{1^n} \rangle$: Can-Loehr

$\nabla(s_{\text{hook}})$ → Labeled Dyck paths with late first return

Loehr-W

$\nabla(s_\lambda)$ → Labeled, nested Dyck paths

Loehr-W

$\nabla(s_\lambda)$

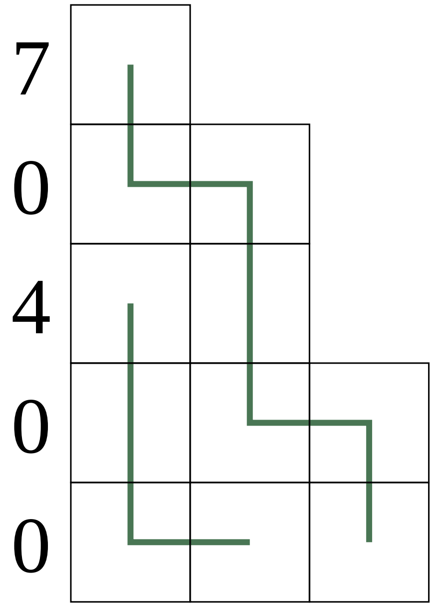
Conjecture (Loehr-W)**.** For any partition $\boldsymbol{\lambda}$,

$$\nabla(s_\lambda) = \mathrm{sgn}(\lambda) \sum_{(\Pi,R)\in LNDP_\lambda} t^{\mathrm{area}(\Pi,R)} q^{\mathrm{dinv}(\Pi,R)} x_R,$$

where

- $\Pi = (\pi_0, \ldots, \pi_{\ell(\lambda')-1})$ is a tuple of **N**ested **D**yck **P**aths
- $R = (r_0, \ldots, r_{\ell(\lambda')-1})$ is a tuple of **L**abels.

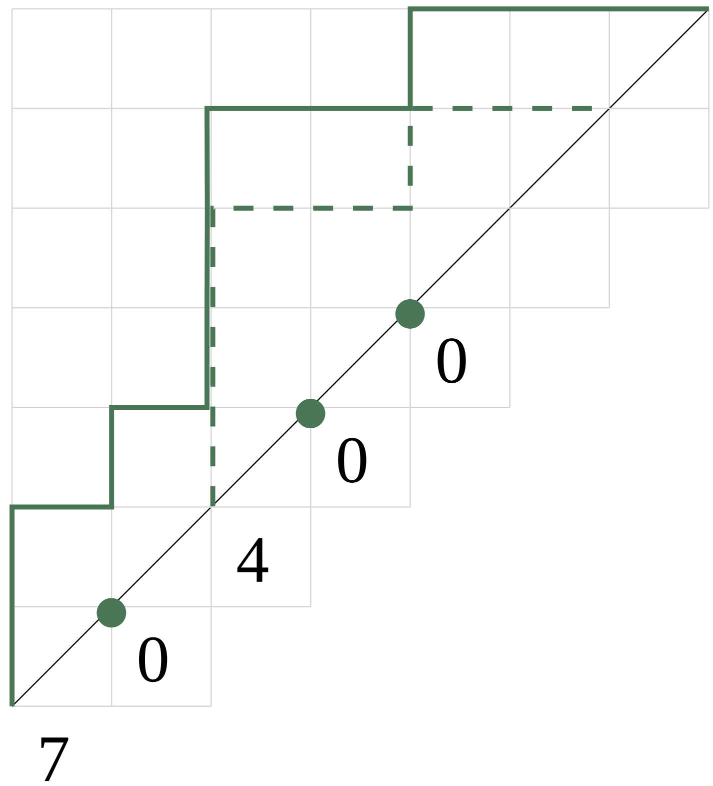
One term of $\nabla(s_{542})|_{m_{111}}$

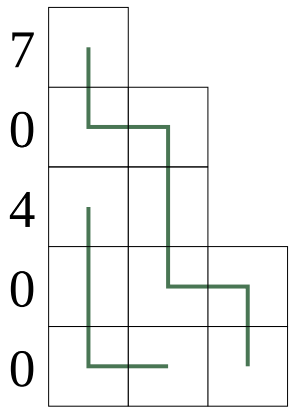


Sign depends on number of rows crossed by rim hooks.

Lengths of rim hooks determine lengths of Dyck paths.

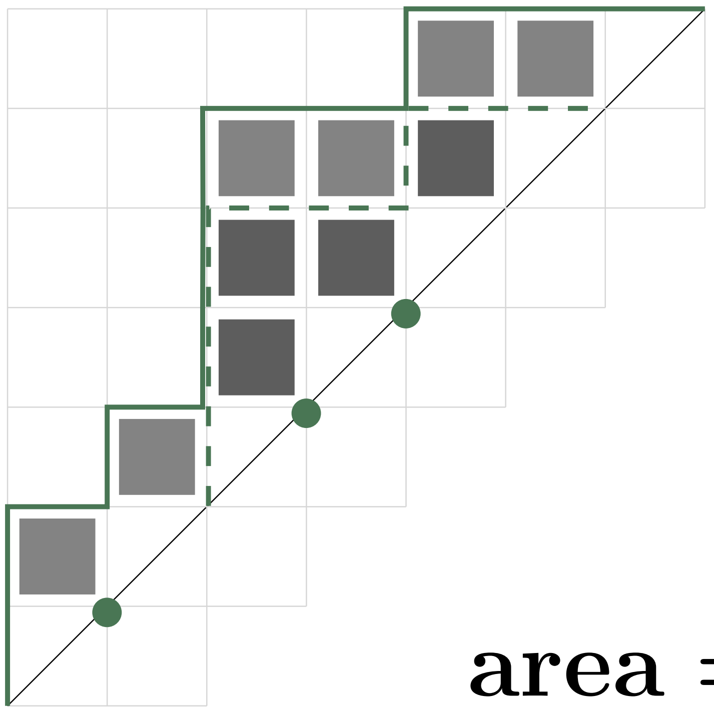
One term of $\nabla(s_{542})|_{m_{111}}$





$$(-1)^6 t^{\square} q^{\square} \square$$

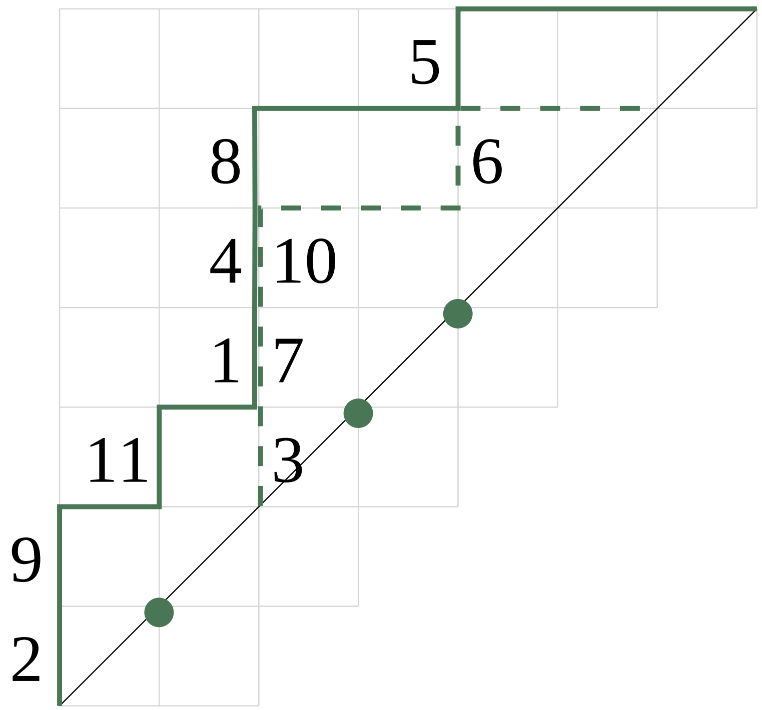
One term of $\nabla(s_{542})\big|_{m_{111}}$



$$\text{area} = 6 \cdot 1 + 4 \cdot 2$$

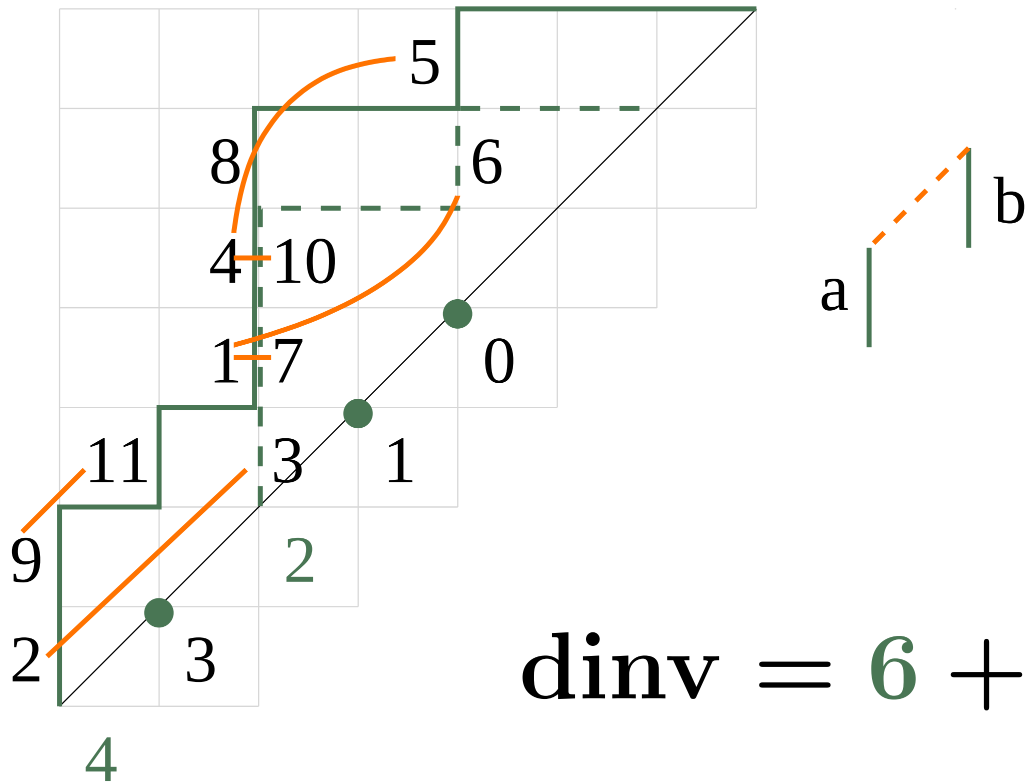
$$(-1)^6 t^{14} q^{\square} \square$$

One term of $\nabla(s_{542})|_{m_{1^{11}}}$



$$(-1)^6 t^{14} q^{\square} x_1 x_2 \cdots x_{11}$$

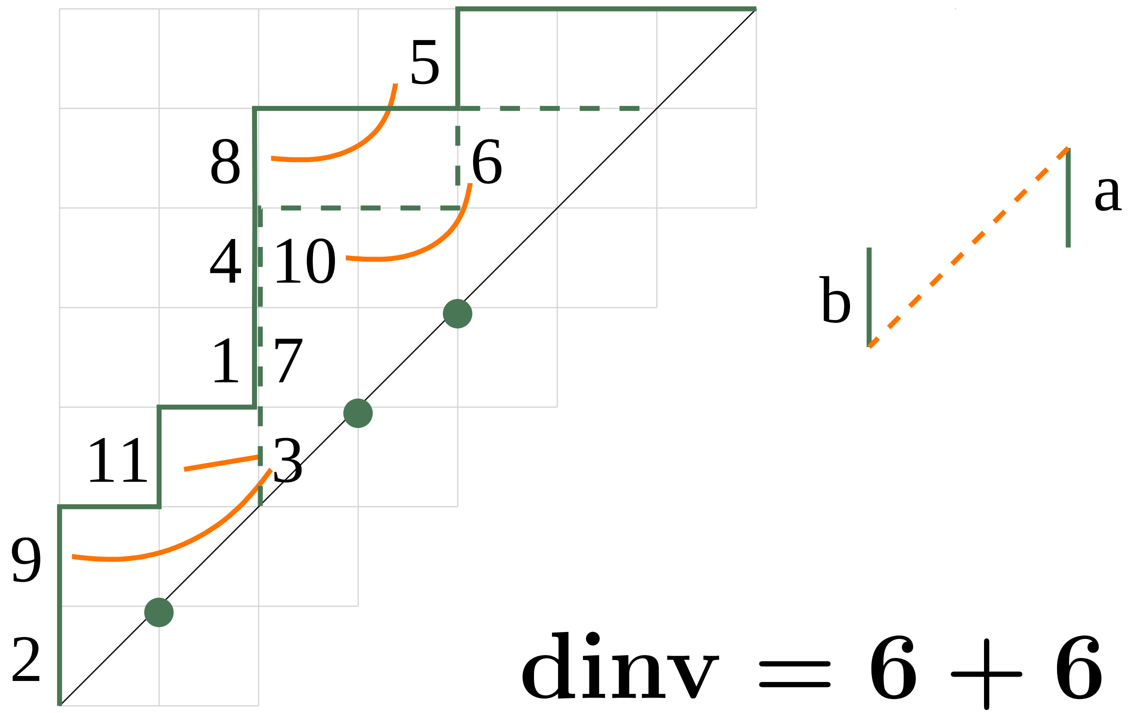
One term of $\nabla(s_{542})\big|_{m_{1^{11}}}$



$$\mathbf{dinv} = 6 + 6 +$$

$$(-1)^6 t^{14} q^{\square} x_1 x_2 \cdots x_{11}$$

One term of $\nabla(s_{542})|_{m_{1^{11}}}$



$$\mathbf{dinv} = 6 + 6 + 4$$

$$(-1)^6 t^{14} q^{16} x_1 x_2 \cdots x_{11}$$

$LNDP_\lambda$

$LNDP_\lambda = \{(\Pi, R)\}$ as before such that

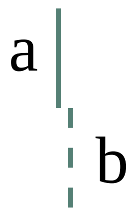
- The i-th path in Π starts at (i, i) and has length equal to that of the i-th hook from the top.

- The entries in i-th label vector in R strictly increase up columns of corresponding path.

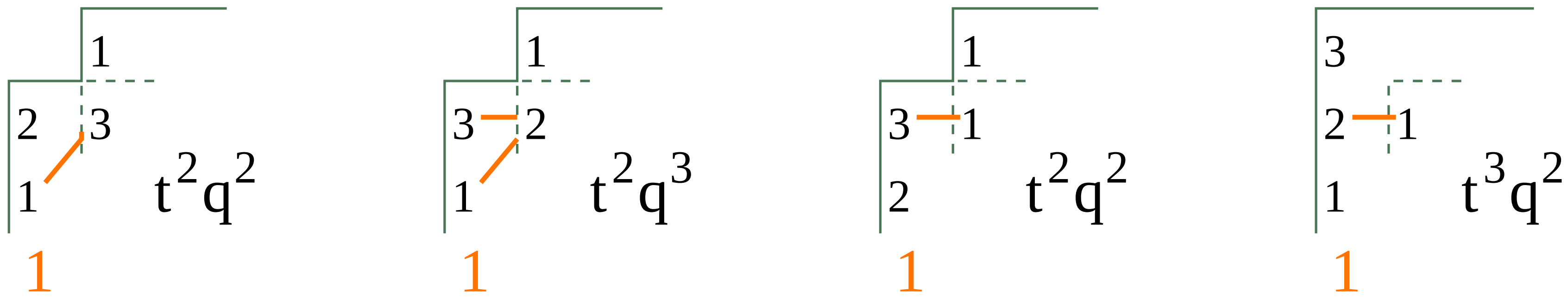
And furthermore. . .

Paths can't

- cross
- share east edges
- pass through another's start
- have $a > b$:



Coefficient of m_{211} in $\nabla(s_{22})$

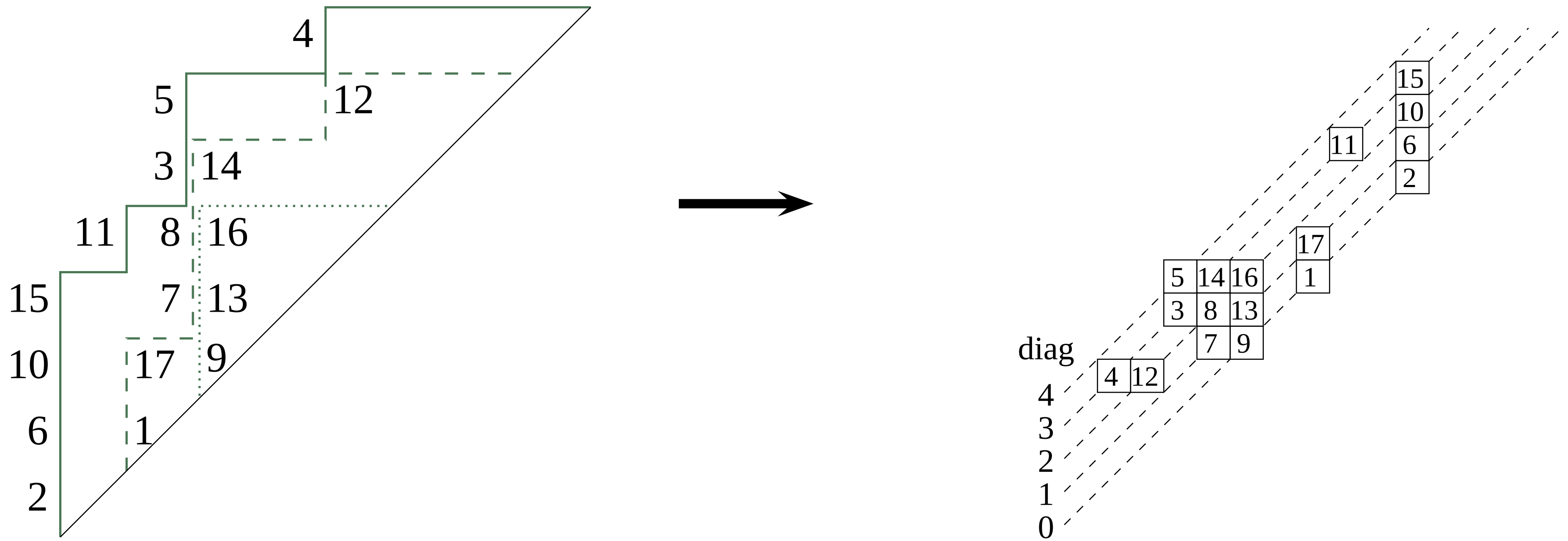


$$\nabla(s_{22}) = -t^2q^2m_{31} - t^2q^2m_{22}$$

$$-t^2q^2(2+t+q)m_{211}$$

$$-t^2q^2(3t+3q+3+tq)m_{1^4}$$

LLT Polynomials



LLT Polynomials

$$\sum_{R:\ (\Pi,R)\in LNDP_\lambda} q^{\mathrm{dinv}(\Pi,R)} x_R = q^{\mathrm{adj}(\lambda)+n(\Gamma(\Pi))} \sum_{T\in SSYT_{\Gamma(\Pi)}} q^{\mathrm{dinv}(T)} x_T$$