Quasisymmetric and Schur expansions of cycle index polynomials

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Abstract

Given a subgroup G of the symmetric group S_n, the cycle index polynomial cyc_G is the average of the power-sum symmetric polynomials indexed by the cycle types of permutations in G. By Pólya's Theorem, the monomial expansion of cyc_G is the generating function for weighted colorings of n objects, where we identify colorings related by one of the symmetries in G. This paper develops combinatorial formulas for the fundamental quasisymmetric expansions and Schur expansions of certain cycle index polynomials. We give explicit bijective proofs based on standardization algorithms applied to equivalence classes of colorings. Subgroups studied here include Young subgroups of S_n, the alternating groups A_n, direct products, conjugate subgroups, and certain cyclic subgroups of S_n generated by $(1, 2, \dots, k)$. The analysis of these cyclic subgroups when k is prime reveals an unexpected connection to perfect matchings on a hypercube with certain vertices identified.

Keywords: cycle index polynomials; fundamental quasisymmetric polynomials; Schur symmetric polynomials; weighted colorings; Pólya's Theorem; perfect matchings.

1. Introduction

1.1. Cycle Index Polynomials

Given a subgroup G of the symmetric group S_n, the *cycle index polynomial* cyc_G is the average of the power-sum symmetric polynomials indexed by the cycle types of permutations in G. (See §2 for precise definitons of these terms and other notation used in the Introduction.) Pólya's Theorem states that the individual monomials in cyc_G correspond to weighted colorings of n objects, where we identify colorings related by one of the symmetries in G.

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Like every symmetric polynomial, cyc_G has an expansion in terms of the Schur basis $\{s_\lambda\}$: for each integer partition λ of n, there is a unique scalar $c(G,\lambda)$ with $\mathrm{cyc}_G = \sum_{\lambda\in\mathrm{Par}(n)} c(G,\lambda)s_\lambda$. As explained below, the coefficients $c(G,\lambda)$ are nonnegative integers that give the multiplicities of the irreducible constituents in a certain linear representation of S_n. One main goal of this paper is to develop concrete combinatorial formulas for these coefficients for certain choices of G, and to prove these formulas via explicit bijections involving weighted colorings. For example, Theorem 19 states that $c(G,\lambda)$ counts the number of standard tableaux of shape λ whose descent set lies in a prescribed collection $\mathcal{D}$ that depends on G.

Since finding combinatorial expressions for Schur coefficients is often difficult, another possibility is to write cyc_G in terms of Gessel's fundamental basis $\{F_{n,S}\}$ for the space of quaisisymmetric polynomials [9]. In this case, for each subset S of $\{1,2,\ldots,n-1\}$ there is a unique scalar $b(G,S)$ with $\mathrm{cyc}_G = \sum_S b(G,S)F_{n,S}$. A second goal of this paper is to give bijective proofs of combinatorial formulas for the coefficients $b(G,S)$ for many choices of the subgroup G. Here our typical formula looks like

$$\mathrm{cyc}_G(x_1,\ldots,x_m) = \sum_{s\in\mathcal{C}} F_{n,\mathrm{IDes}(s)}(x_1,\ldots,x_m) \tag{1}$$

where $\mathcal{C}$ is an explicit subset of S_n determined by G.

Finding F-expansions is a stepping-stone that gets us closer to the Schur expansion compared to the original formulas involving power-sums or individual monomials. In our specific problem, when $\mathcal{C}$ has the particular form

$$\mathcal{C} = \{s\in S_n : \mathrm{Des}(s)\in\mathcal{D}\} \tag{2}$$

for some collection $\mathcal{D}$, then we are able to find Schur expansions via Theorem 19. More generally, the combinatorial importance of F-expansions has become increasingly prominent in the recent literature on Macdonald polynomials, Lascoux–Leclerc–Thibon (LLT) polynomials, diagonal harmonics, and Foulkes's Conjecture; see, for instance, [12, 13, 14, 15, 21]. Egge and the present authors [7] have given a combinatorial method for automatically converting any F-expansion into a Schur expansion, although the resulting coefficients have mixed signs in general. In an unpublished manuscript, Garsia and Remmel gave an algebraic reformulation of the result in [7], which was recently proved combinatorially by Gessel [10]. Assaf [2] has also developed powerful machinery for combinatorially proving Schur positivity when certain axioms are satisfied.

We briefly mention some other research that highlights the growing significance of quasisymmetric functions in modern algebraic combinatorics. Richard Stanley used quasisymmetric functions in a crucial way in his enumeration of reduced words for the longest word in a Coxeter group [33] and his study of riffle shuffles [32]. Stanley's chromatic symmetric functions [31] have been generalized to chromatic quasisymmetric functions [3, 16, 29]. Some informative quasisymmetric expansions appear in [4, 8, 19, 24]. Sums of fundamental quasisymmetric polynomials over sets of permutations are considered in such works as [1, 6, 11].

1.2. Motivation from Representation Theory

To understand why the Schur coefficients of cyc_G are so interesting, we must recall the representation-theoretical significance of the cycle index polynomial. For any subgroup G of S_n, let S_n/G be the set of left cosets of G in S_n, and let $\mathbb{C}[S_n/G]$ be the $\mathbb{C}$-vector space with basis S_n/G. S_n acts on the set S_n/G by left multiplication, so $\mathbb{C}[S_n/G]$ becomes an S_n-module. It is well-known [23, I.7 Ex. 4, pg. 117] that the symmetric polynomial cyc_G is the Frobenius characteristic of this module. In other words, the coefficient $c(G, \lambda)$ of s_λ in cyc_G is the multiplicity of the irreducible module indexed by λ in $\mathbb{C}[S_n/G]$.

More generally, suppose S_n acts on any finite set X. The S_n-module $\mathbb{C}[X]$ is a direct sum of submodules $\mathbb{C}[Y]$ corresponding to the orbits Y of the action. Each submodule $\mathbb{C}[Y]$ is isomorphic to $\mathbb{C}[S_n/G]$, where G is the stabilizer of any point in Y. Thus, we can find the character of an arbitrary linear representation arising from a group action of S_n if we know all the coefficients $c(G, \lambda)$ of the relevant stabilizer subgroups G.

On one hand, for any specific subgroup G, the Schur coefficients can be found algebraically by taking the inner product of the character of $\mathbb{C}[S_n/G]$ with irreducible characters. Alternatively, we can pass from the known monomial expansion or power-sum expansion of cyc_G to the Schur expansion by multiplying by the appropriate transition matrix (see [25, §6.2] for examples of computations using this technique). On the other hand, the required transition matrices (the inverse Kostka matrix in the case of the monomial expansion, or the character table of S_n in the case of the power-sum expansion) both have coefficients of mixed sign. So the resulting algebraic formulas for the Schur coefficients are complicated combinations of positive and negative objects, which are not manifestly positive (or even integral, if we start with the power-sum expansion, which involves division by $|G|$). Thus, these algebraic solutions are not satisfactory from the combinatorial viewpoint. Here we are asking for bijective proofs that the Schur coefficients count explicitly identifiable tableau-like structures. This combinatorial question is unsolved for general G, although the case of Young subgroups is treated in many representation theory textbooks (see, e.g., [28, §2.11]). In that case, the Schur coefficients are the famous *Kostka numbers*, which count semistandard tableaux of a given shape and content.

The Foulkes Conjecture can be phrased in terms of the coefficients $c(G, \lambda)$. Let $X_{a,b}$ be the set of set partitions of $\{1, 2, \ldots, ab\}$ into a blocks of size b. S_{ab} acts transitively on $X_{a,b}$ in the obvious way; let $G_{a,b}$ be the stabilizer subgroup of a designated element of $X_{a,b}$. The Foulkes Conjecture states that if $a \leq b$ then for all λ, $c(G_{a,b}, \lambda) \leq c(G_{b,a}, \lambda)$. Knowing combinatorial formulas for these coefficients could help prove the inequality. The present authors found explicit F-expansions for the cycle index polynomials $\mathrm{cyc}_{G_{a,b}}$ in [21].

1.3. Summary of New Results

The primary contributions of this paper are as follows. First, we describe a method (called the *standardization approach*) that provides uniform bijective proofs for F-expansions of the form (1) for many choices of the subgroup

G. When the collection $\mathcal{C}$ has the special form (2), we also obtain bijective proofs of the Schur expansions with the aid of the Robinson–Schensted correspondence. Standardization is a powerful and well-known construction for studying quasisymmetric expansions that has been utilized in many papers including [5, 11, 21, 29]. The novel issue pursued here is the delicate interaction between the standardization map and equivalence classes of words induced by the action of G.

We apply our standardization approach to analyze the alternating groups A_n, Young subgroups of S_n, direct products of subgroups, conjugate subgroups, and all subgroups of S_n for $n \leq 5$ (among other examples). The approach succeeds for many but not all subgroups G. Another major contribution is a complete solution of the problem for the subgroups $G = \langle(1, 2, \ldots, p)\rangle \leq S_n$, where p is any odd prime (Theorems 22 and 24). This analysis reveals an unexpected connection between the standardization process and the existence of perfect matchings in a certain quotient of a hypercube graph.

1.4. Outline of Paper

The rest of this paper is structured as follows. §2 presents the necessary background material. §3 recalls the sorting and standardization maps, introduces the standardization approach, and illustrates the approach by finding F-expansions for $G = \{\mathrm{id}_n\}$, $G = S_n$, and $G = A_n$. §4 studies subgroups G that are direct products or conjugates of subgroups that have already been solved. We deduce formulas for the cycle index polynomials of Young subgroups and explain what to do when a subgroup G of S_n is embedded in a larger symmetric group. The cyclic subgroup $G = \langle(1, 2, 3, 4)\rangle$ of S_n and an 8-element dihedral subgroup D of S_n provide interesting examples of the strengths and limitations of our approach. §5 shows how to pass from the F-expansion to the Schur expansion when (2) holds. §6 uses the standardization approach to analyze the cyclic subgroup $\langle(1, 2, \ldots, p)\rangle$ of S_n where p is an odd prime. Finally, §7 contains some further results and conjectures based on computer-generated data, including an analysis of all subgroups of S_n for $n \leq 5$.

2. Background

We assume readers are familiar with basic facts and notation regarding permutations, integer partitions, and symmetric polynomials, which can be found in references such as [20, 28].

2.1. Cycle Index Polynomials

For each positive integer n, let $[n] = \{1, 2, \ldots, n\}$. Let S_n be the symmetric group on n symbols, which is the group of all bijections $g : [n] \to [n]$ under the operation of composition of functions. Each $g \in S_n$ can be factored into a product of disjoint cycles. Define the *cycle type* of g, denoted $\mathrm{type}(g)$, to be the integer partition of n obtained by listing the lengths of all the cycles of g, including any 1-cycles, in weakly decreasing order. For example, $g = (1,2,3,9)(5,7)(6,8)(4) \in S_9$ has $\mathrm{type}(g) = (4,2,2,1)$.

Given positive integers k and m, the k^{th} *power-sum symmetric polynomial in m variables* is $p_k = p_k(x_1,\ldots,x_m) = x_1^k + x_2^k + \cdots + x_m^k$. For an integer partition $\mu = (\mu_1,\mu_2,\ldots,\mu_s)$, define $p_\mu = p_{\mu_1}p_{\mu_2}\cdots p_{\mu_s}$. Given any subgroup G of S_n, define the *cycle index* of G to be the symmetric polynomial

$$\mathrm{cyc}_G(x_1,\ldots,x_m) = \frac{1}{|G|}\sum_{g\in G} p_{\mathrm{type}(g)}(x_1,\ldots,x_m). \tag{3}$$

For example, the cyclic subgroup $G = \langle(1,2,3,4)\rangle \subseteq S_4$ has

$$\mathrm{cyc}_G = \frac{1}{4}(p_{(1,1,1,1)} + 2p_{(4)} + p_{(2,2)}).$$

2.2. Colorings with Symmetries

Fix positive integers m and n. A *coloring* of the set $[n]$ using m available colors is a function $w : [n] \to [m]$. We identify the function w with the word $w_1w_2\cdots w_n \in [m]^n$, where $w_i = w(i) \in [m]$ is the color assigned to position i. The *weight* of w is $\mathrm{wt}(w) = x_{w_1}x_{w_2}\cdots x_{w_n}$. Let $W = [m]^n$ be the set of all weighted colorings. One readily checks that

$$\sum_{w\in W}\mathrm{wt}(w) = (x_1+\cdots+x_m)^n = p_{(1^n)}(x_1,\ldots,x_m) = \mathrm{cyc}_{\{\mathrm{id}_n\}}(x_1,\ldots,x_m),$$

where $\{\mathrm{id}_n\} = \{(1)(2)\cdots(n)\}$ is the identity subgroup of S_n.

Now suppose G is a given subgroup of S_n. The group G acts on the set W via the rule $g \star w = w \circ g^{-1}$ for $g \in G$ and $w \in W$. This action decomposes W into a disjoint union of orbits. For $w \in W$, let $[w]_G = \{g \star w : g \in G\}$ be the orbit of w under G. Intuitively, the orbit $[w]_G$ consists of all colorings that get identified with w when the symmetries in G are taken into account. Each coloring in the orbit of w is obtained from w by rearranging positions (inputs to the function w) using one of the allowed symmetries in G. This rearrangement does not change the multiset of colors used, so $\mathrm{wt}(w) = \mathrm{wt}(v)$ for all $v \in [w]_G$. Thus we can define the weight of an orbit by $\mathrm{wt}([w]_G) = \mathrm{wt}(w)$ for all $w \in W$.

For example, suppose $m = n = 4$ and $G = \langle(1,2,3,4)\rangle$. The orbit $[1312]_G = \{1312, 3121, 1213, 2131\}$ has size 4 and weight $x_1^2x_2x_3$, and the orbit $[2424]_G = \{2424, 4242\}$ has size 2 and weight $x_2^2x_4^2$. Since G acts by cyclically shifting positions, we can think of each orbit $[w]_G$ as a 4-bead *necklace*, where all rotations of a given necklace are considered the same.

Returning to the general case, let $W/G = \{[w]_G : w \in W\}$ be the set of all orbits of the action of G on W. The following celebrated theorem of Pólya provides the expansion of the cycle index polynomial into individual monomials. This theorem says that cyc_G is the generating function for the weighted set of colorings W/G, where colorings related by symmetries in G have been identified.

Theorem 1 (Pólya [26, 27]). *For any subgroup G of S_n and all integers $m > 0$,*

$$\mathrm{cyc}_G(x_1,\ldots,x_m) = \sum_{[w]_G \in [m]^n/G} \mathrm{wt}([w]_G). \tag{4}$$

For a proof, see [25, Theorem 6.1] or [20, §7.16].

2.3. Fundamental Quasisymmetric Polynomials

Given any subset S of $[n-1]$, let $W_{n,S}$ be the set of all words $w = w_1w_2\cdots w_n$ in $[m]^n$ such that $w_1 \le w_2 \le \cdots \le w_n$ and for all $j \in S$, $w_j < w_{j+1}$. So $W_{n,S}$ is the set of weakly increasing words in W that have strict increases at all the positions in S (and perhaps elsewhere). Gessel's *fundamental quasisymmetric polynomial* indexed by n and S is defined to be

$$F_{n,S}(x_1,\ldots,x_m) = \sum_{w \in W_{n,S}} \mathrm{wt}(w). \tag{5}$$

For example, taking $m = n = 4$, $W_{4,\{1,3\}} = \{1223, 1224, 1334, 2334, 1234\}$ and

$$F_{4,\{1,3\}}(x_1,x_2,x_3,x_4) = x_1x_2^2x_3 + x_1x_2^2x_4 + x_1x_3^2x_4 + x_2x_3^2x_4 + x_1x_2x_3x_4.$$

Some authors use different notation for $F_{n,S}$. In particular, these polynomials are often indexed by compositions of n rather than pairs (n,S) with $S \subseteq [n-1]$.

In applications, the set S indexing $F_{n,S}$ is often the inverse descent set of a permutation. For any $s \in S_n$ written as a word $s_1s_2\cdots s_n$, the *descent set* of s is $\mathrm{Des}(s) = \{i < n : s_i > s_{i+1}\}$, which is a subset of $[n-1]$. The *inverse descent set* of $s \in S_n$ is $\mathrm{IDes}(s) = \mathrm{Des}(s^{-1})$, where s^{-1} is the inverse of s in the group S_n. Equivalently, one readily checks that $\mathrm{IDes}(s)$ is the set of all $i < n$ such that $i+1$ appears to the left of i in the word $s_1s_2\cdots s_n$. For example, $s = 435612 \in S_6$ has $\mathrm{Des}(s) = \{1,4\}$, $s^{-1} = 562134$, and $\mathrm{IDes}(s) = \mathrm{Des}(s^{-1}) = \{2,3\}$.

We now recall the combinatorial rule for the F-expansion of the product of fundamental quasisymmetric polynomials [22, p. 35]. Suppose $s^{(i)} \in S_{n_i}$ for $1 \le i \le d$. Let $N = \sum_i n_i$, $N_i = \sum_{j<i} n_j$, and let $\overline{s}^{(i)}$ be the word obtained from $s^{(i)}$ by adding N_i to each letter. A permutation $u \in S_N$ is called a *shuffle* of $s^{(1)},\ldots,s^{(d)}$ iff all of the words $\overline{s}^{(i)}$ appear as subsequences of the word $u = u_1u_2\cdots u_N$. For example, $u = 53614782$ is a shuffle of $s^{(1)} = 312$, $s^{(2)} = 231$, and $s^{(3)} = 12$; here $\overline{s}^{(1)} = 312$, $\overline{s}^{(2)} = 564$, and $\overline{s}^{(3)} = 78$.

Proposition 2. *With the above terminology,*

$$\prod_{i=1}^{d} F_{n_i,\mathrm{Des}(s_i)}(x_1,\ldots,x_m) = \sum_{\substack{u \in S_n:\ u \text{ is a shuffle} \\ \text{of } s_1,\ldots,s_d}} F_{N,\mathrm{Des}(u)}(x_1,\ldots,x_m). \tag{6}$$

2.4. Schur Symmetric Polynomials

Next we recall two combinatorial definitions of the Schur symmetric polynomials. Let $\mathrm{Par}(n)$ be the set of integer partitions of n. Given $\lambda = (\lambda_1, \ldots, \lambda_s)$ in $\mathrm{Par}(n)$, the *diagram* of λ (in English notation) is an array of unit cells with λ_i left-justified cells in the i^{th} row from the top. A *semistandard (Young) tableau* T of shape λ using the alphabet $[m]$ is a filling of the cells in the diagram of λ with values in $[m]$ such that the entries in each row weakly increase from left to right, and the entries in each column strictly increase from top to bottom. The *weight* of such a tableau T is $\mathrm{wt}(T) = x_1^{e_1} \cdots x_m^{e_m}$, where e_j is the number of occurrences of the symbol j in T. We write $\mathrm{SSYT}_m(\lambda)$ for the set of semistandard tableaux of shape λ with entries in $[m]$. The *Schur symmetric polynomial* indexed by λ in m variables is

$$s_\lambda(x_1, \ldots, x_m) = \sum_{T \in \mathrm{SSYT}_m(\lambda)} \mathrm{wt}(T).$$

Schur polynomials have a nice expansion in terms of the fundamental quasisymmetric polynomials. Suppose λ is a partition of n. A *standard tableau* U of shape λ is a tableau in $\mathrm{SSYT}_m(\lambda)$ with weight $x_1 x_2 \cdots x_n$; this means that the numbers $1, 2, \ldots, n$ appear once each in U. The *descent set* of U is the set of $i < n$ such that $i + 1$ appears in U in a lower row than i. Let $\mathrm{SYT}(\lambda)$ be the set of all standard tableaux of shape λ. The following formula of Gessel [9] can be proved bijectively by standardizing reading words of semistandard tableaux (see [20, Thm. 12.99]):

$$s_\lambda(x_1, \ldots, x_m) = \sum_{U \in \mathrm{SYT}(\lambda)} F_{n,\mathrm{Des}(U)}(x_1, \ldots, x_m). \tag{7}$$

For example, given the partition $\lambda = (3, 2) \in \mathrm{Par}(5)$, we compute

$$\mathrm{SYT}(\lambda) = \left\{ \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} \end{array}, \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2} \end{array}, \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} \end{array}, \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2} \end{array}, \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} \end{array} \right\};$$

$$s_{(3,2)} = F_{5,\{3\}} + F_{5,\{2,4\}} + F_{5,\{2\}} + F_{5,\{1,4\}} + F_{5,\{1,3\}}.$$

3. The Standardization Approach

This section describes the *standardization approach*, which is the main tool we use to obtain bijective proofs of the formulas (1) for various groups G.

3.1. Sorting and Standardization

Fix positive integers m and n, and let $W = [m]^n$ be the set of words of length n using the alphabet $[m]$. We recall the definitions of two maps on W called *sorting* and *standardization*. Given a word $w \in W$, $\mathrm{sort}(w)$ is the word obtained by sorting the letters of w into weakly increasing order. Next we define the standardization map $\mathrm{stdz} : W \to S_n$. To standardize a word

$w \in W$ containing n_1 ones, n_2 twos, etc., renumber the n_1 ones in w, from left to right, with the symbols $1, 2, \ldots, n_1$. Then renumber the n_2 twos originally in w, from left to right, with the symbols $n_1+1, n_1+2, \ldots, n_1+n_2$. Continue similarly; in general, the n_i copies of i in w get renumbered from left to right with the symbols $n_1+\cdots+n_{i-1}+1, n_1+\cdots+n_{i-1}+2, \ldots, n_1+\cdots+n_i$. For example, if $m = n = 9$ and $w = 311421223 \in W$, then $\mathrm{sort}(w) = 111222334$ and $\mathrm{stdz}(w) = 712943568 \in S_9$.

For any $s \in S_n$, let $U(s) = \{w \in W : \mathrm{stdz}(w) = s\}$ be the set of all words in W that standardize to s. For example, $U(4213) = \{3212, 4212, 4313, 4323, 4213\}$. Shrinking the domain and codomain of the sorting function gives a restricted map $\mathrm{sort}_s : U(s) \to W_{n,\mathrm{IDes}(s)}$ that is easily seen to be a weight-preserving bijection. The two-sided inverse of sort_s is the map $\mathrm{unsort}_s : W_{n,\mathrm{IDes}(s)} \to U(s)$ that sends $v \in W_{n,\mathrm{IDes}(s)}$ to $v_{s_1} v_{s_2} \cdots v_{s_n}$. In other words, unsort_s replaces each symbol j in the word s by v_j. For example, if $s = 712943568$ (so $\mathrm{IDes}(s) = \{3, 6, 8\}$) and $v = 233444567 \in W_{9,\{3,6,8\}}$, then $\mathrm{unsort}_s(v) = 523743446 \in U(s)$. We can now restate (5) as follows.

Proposition 3. *For all $s \in S_n$,*

$$\sum_{w \in U(s)} \mathrm{wt}(w) = F_{n,\mathrm{IDes}(s)}(x_1, \ldots, x_m). \tag{8}$$

3.2. The Standardization Approach

For each $s \in S_n$, Proposition 3 gives a bijective proof that $F_{n,\mathrm{IDes}(s)}$ is the generating function for the weighted set $U(s)$. More generally, for any $\mathcal{C} \subseteq S_n$, define

$$U(\mathcal{C}) = \{w \in W : \mathrm{stdz}(w) \in \mathcal{C}\} = \bigcup_{s \in \mathcal{C}} U(s).$$

Since $U(\mathcal{C})$ is the disjoint union of the sets $U(s)$ for $s \in \mathcal{C}$,

$$\sum_{w \in U(\mathcal{C})} \mathrm{wt}(w) = \sum_{s \in \mathcal{C}} F_{n,\mathrm{IDes}(s)}(x_1, \ldots, x_m). \tag{9}$$

Combining the individual bijections from Proposition 3, we obtain a bijective proof of this formula. Specifically, the weight-preserving bijection sends $w \in U(\mathcal{C})$ to the pair $(\mathrm{stdz}(w), \mathrm{sort}(w))$, where $s = \mathrm{stdz}(w)$ is in $\mathcal{C}$ and $v = \mathrm{sort}(w)$ is in $W_{n,\mathrm{IDes}(s)}$. The inverse bijection sends (s, v) with $s \in \mathcal{C}$ and $v \in W_{n,\mathrm{IDes}(s)}$ to $\mathrm{unsort}_s(v) \in U(\mathcal{C})$.

The next theorem formally states the *standardization approach* for finding F-expansions of cycle index polynomials.

Theorem 4. *Let G be a subgroup of S_n. If there exists $\mathcal{C} \subseteq S_n$ such that*

$$U(\mathcal{C}) \text{ intersects every orbit } [z]_G \in W/G \text{ in exactly one point,} \tag{10}$$

then $\mathrm{cyc}_G = \sum_{s \in \mathcal{C}} F_{n,\mathrm{IDes}(s)}$ holds via a bijective proof.

Proof. Recall that the weight of an orbit $[z]_G$ is the weight of any representative of this orbit. The assumption (10) means that $U(\mathcal{C})$ is a system of distinct representatives for all the orbits. Now using (4) and (9), it follows that

$$\mathrm{cyc}_G = \sum_{[z]_G \in W/G} \mathrm{wt}([z]_G) = \sum_{w \in U(\mathcal{C})} \mathrm{wt}(w) = \sum_{s \in \mathcal{C}} F_{n,\mathrm{IDes}(s)}.$$

The bijection acts on an orbit $[z]_G$ by first finding the unique representative $w \in [z]_G \cap U(\mathcal{C})$, then mapping $[z]_G$ to $(\mathrm{stdz}(w), \mathrm{sort}(w))$. The inverse map sends (s, v) to $[\mathrm{unsort}_s(v)]_G$. □

Definition 5. Given a subgroup G of S_n, we say that the standardization approach *succeeds for* G iff there exists $\mathcal{C} \subseteq S_n$ satisfying (10). We say the standardization approach *fully succeeds* iff there exists a collection $\mathcal{D}$ of subsets of $[n-1]$ such that $\mathcal{C} = \{s \in S_n : \mathrm{Des}(s) \in \mathcal{D}\}$ satisfies (10).

To illustrate this terminology, consider the identity subgroup $G = \{\mathrm{id}_n\}$. One readily sees that (10) holds for $\mathcal{C} = S_n$, since $U(\mathcal{C}) = W$ and every orbit of W/G consists of a single word. We can take $\mathcal{D}$ to be the collection of all subsets of $[n-1]$ here, so the standardization approach fully succeeds for $\{\mathrm{id}_n\}$. Theorem 4 gives a bijective proof of the expansion

$$h_{(1^n)} = p_{(1^n)} = \mathrm{cyc}_{\{\mathrm{id}_n\}} = \sum_{s \in S_n} F_{n,\mathrm{IDes}(s)}(x_1, \ldots, x_m).$$

As another simple example, consider the subgroup $G = S_n$. One readily checks that (10) holds for $\mathcal{C} = \{\mathrm{id}_n\}$; the key point is that each orbit in W/S_n contains exactly one weakly increasing word. So we have a bijective proof that

$$\mathrm{cyc}_{S_n}(x_1, \ldots, x_m) = F_{n,\emptyset}(x_1, \ldots, x_m) = h_n(x_1, \ldots, x_m). \tag{11}$$

We may take $\mathcal{D} = \{\emptyset\}$ here to see that the standardization approach fully succeeds for $G = S_n$. This example is generalized to the case of Young subgroups of S_n in §4.2.

3.3. Analysis of the Alternating Group

For $s = s_1 s_2 \cdots s_n \in S_n$, an *inversion* of s is a pair (i, j) with $1 \le i < j \le n$ and $s_i > s_j$. Let $\mathrm{inv}(s)$ be the number of inversions of s, and define $\mathrm{sgn}(s) = (-1)^{\mathrm{inv}(s)}$. The *alternating group on* $[n]$ is $A_n = \{s \in S_n : \mathrm{sgn}(s) = +1\}$, which is a normal subgroup of S_n. For example, $s = 42513$ has $\mathrm{inv}(s) = 6$ and $\mathrm{sgn}(s) = +1$, so $s \in A_5$. The next result shows that when computing cyc_{A_n}, the standardization approach fully succeeds for $n \bmod 4 \in \{2, 3\}$, but the approach does not succeed for $n \bmod 4 \in \{0, 1\}$. Fortunately, a modified bijection can be found in the latter case.

Theorem 6. *For all integers n such that $n \bmod 4 \in \{2,3\}$, the standardization approach bijectively proves that*

$$\begin{aligned}\mathrm{cyc}_{A_n}(x_1,\ldots,x_m) &= F_{n,\emptyset}(x_1,\ldots,x_m) + F_{n,[n-1]}(x_1,\ldots,x_m)\\ &= \sum_{s\in S_n\colon \mathrm{Des}(s)\in\{\emptyset,[n-1]\}} F_{n,\mathrm{IDes}(s)}(x_1,\ldots,x_m).\end{aligned}$$

For all $n > 1$ with $n \bmod 4 \in \{0,1\}$, the same formula holds via a different bijection.

Proof. Let $r \in S_n$ be the *reverse* permutation $n, n-1, \ldots, 3, 2, 1$; note that $\mathrm{inv}(r) = \binom{n}{2}$. First assume $n \bmod 4 \in \{2,3\}$, which holds iff $\binom{n}{2} \bmod 2 = 1$ iff $\mathrm{sgn}(r) = -1$. We show that (10) holds if we take $\mathcal{C} = \{\mathrm{id}_n, r\} \subseteq S_n$, which has the form (2) for $\mathcal{D} = \{\emptyset, [n-1]\}$. For any word $w \in W$, $w \in U(\mathrm{id}_n)$ iff $\mathrm{stdz}(w) = \mathrm{id}_n$ iff w is a weakly increasing word, whereas $w \in U(r)$ iff $\mathrm{stdz}(w) = r$ iff w is a strictly decreasing word. So

$$U(\mathcal{C}) = \{w \in W : w \text{ is weakly increasing or strictly decreasing}\}.$$

To check (10), consider an arbitrary orbit $[z]_{A_n}$ where $z \in W$. Case 1: z contains a repeated letter, say $z_i = z_j$ with $i \neq j$. We claim that there exists $g \in A_n$ such that $g \star z = \mathrm{sort}(z)$, which is weakly increasing. There certainly exists $h \in S_n$ with $h \star z = \mathrm{sort}(z)$. If h is in A_n, choose $g = h$. If h is not in A_n, choose $g = h \circ (i,j)$, which is in A_n since $\mathrm{sgn}(g) = \mathrm{sgn}(h)\,\mathrm{sgn}((i,j)) = (-1)^2 = +1$. Because $z_i = z_j$, we have $g \star z = h \star ((i,j) \star z) = h \star z = \mathrm{sort}(z)$, as needed. Now we know that the orbit $[z]_{A_n}$ contains the weakly increasing word $\mathrm{sort}(z)$, which is clearly the only weakly increasing word in this orbit. Also, every word in this orbit is a rearrangement of z, which has a repeated letter, so there is no strictly decreasing word in this orbit. Thus (10) holds in Case 1.

Case 2: All letters of z are distinct. Let $z^+ = \mathrm{sort}(z)$, and let z^- be the reversal of z^+. We claim exactly one of z^+ and z^- is in the A_n-orbit of z. This will suffice to verify (10), since z^+ is the only weakly increasing word that might belong to this orbit, and z^- is the only strictly decreasing word that might belong to this orbit. Because the letters of z are distinct, there exists a *unique* $h \in S_n$ such that $h \star z = z^+$. Since $r \star z^+ = z^-$, $r \circ h$ is the unique permutation in S_n sending z to z^-. Since $\mathrm{sgn}(r) = -1$, exactly one of h and $r \circ h$ is in A_n. Thus, exactly one of z^+ and z^- is in $[z]_{A_n}$, as claimed.

Now suppose $n \bmod 4 \in \{0,1\}$ and $n > 1$. The argument in Case 2 fails for such an n, since $\mathrm{sgn}(h) = \mathrm{sgn}(r \circ h)$, so z^+ and z^- are either both in $[z]_{A_n}$ or both not in $[z]_{A_n}$. But the following modification works instead. Consider the transposition $(1,2) \in S_n$, which is $2134\cdots n$ in word notation and has sign -1. Let $U^*((1,2))$ be the set of $w \in U((1,2))$ where all letters of w are distinct. Equivalently, $U^*((1,2)) = \{w \in [m]^n : w_2 < w_1 < w_3 < w_4 < \cdots < w_n\}$. By repeating the proof in Case 1 and Case 2 above, with r replaced by $(1,2)$ and z^- replaced by $(1,2) \star z^+$, one sees that:

$$U(\mathrm{id}_n) \cup U^*((1,2)) \text{ intersects every orbit } [z]_{A_n} \text{ in exactly one point.} \tag{12}$$

On one hand, Proposition 3 proves bijectively that $\sum_{w\in U(\mathrm{id}_n)} \mathrm{wt}(w) = F_{n,\emptyset}$. On the other hand, by sorting the letters of $w \in U^*((1,2))$, we obtain a bijective proof that

$$\sum_{w\in U^*((1,2))} \mathrm{wt}(w) = \sum_{1\leq i_1<i_2<\cdots<i_n\leq m} x_{i_1}x_{i_2}\cdots x_{i_n} = F_{n,[n-1]}(x_1,\ldots,x_m).$$

Now (12) implies that $\mathrm{cyc}_{A_n} = F_{n,\emptyset} + F_{n,[n-1]}$. The bijection acts as follows. Given $z \in W$, let w be the unique representative of $[z]_{A_n}$ in $U(\mathrm{id}_n) \cup U^*((1,2))$. If $w \in U(\mathrm{id}_n)$, map $[z]_{A_n}$ to (id_n, w). If $w \in U^*((1,2))$, map $[z]_{A_n}$ to $((1,2), \mathrm{sort}(w))$. □

For $s \in S_n$, $\mathrm{IDes}(s) = \emptyset$ iff $s = \mathrm{id}_n$, whereas $\mathrm{IDes}(s) = [n-1]$ iff $s = r$ (the reversal of id_n). Now that we know $\mathrm{cyc}_{A_n} = F_{n,\emptyset} + F_{n,[n-1]}$ for all $n > 1$, it follows that $\mathcal{C} = \{\mathrm{id}_n, r\}$ is the only subset of S_n that could possibly satisfy condition (10) for $G = A_n$. But we saw in the proof above that this set $\mathcal{C}$ does not satisfy the condition when $n \bmod 4 \in \{0,1\}$. This proves that the standardization approach does *not* succeed for these choices of G. Nevertheless, the underlying algebraic formula still holds, and in this case we were able to find a bijective proof by a modification of the standardization approach.

Remark 7. There is an easy representation-theoretic proof of the formula $\mathrm{cyc}_{A_n} = F_{n,\emptyset}+F_{n,[n-1]} = s_{(n)}+s_{(1^n)}$, based on the fact that cyc_{A_n} is the Frobenius characteristic of the S_n-module $\mathbb{C}[S_n/A_n]$. This two-dimensional module is the direct sum of one copy of the trivial representation and one copy of the sign representation, as one readily checks.

4. Direct Products, Conjugate Subgroups, and Compressed Words

This section combines the standardization approach with some general constructions that lead to formulas for cyc_G for more subgroups G.

4.1. Solution for Direct Products

We begin by studying the cycle index polynomial for certain direct products of subgroups. Throughout this subsection, we use the following notation. Suppose G_i is a subgroup of S_{n_i} for $1 \leq i \leq d$. Define $N = n_1 + n_2 + \cdots + n_d$ and $N_i = \sum_{j<i} n_j$ for $1 \leq i \leq d$. Let G_i' be the isomorphic copy of G_i in the group S_N that permutes the symbols $\{1 + N_i, 2 + N_i, \ldots, n_i + N_i\}$ in the same way that G_i permutes the symbols $\{1, 2, \ldots, n_i\}$. Let

$$G = \{g_1' \circ g_2' \circ \cdots \circ g_d' : \ g_i' \in G_i' \text{ for } 1 \leq i \leq d\}.$$

G is a subgroup of S_N, namely the (internal) direct product $G_1' \times G_2' \times \cdots \times G_d'$, which is isomorphic to the (external) direct product $G_1 \times G_2 \times \cdots \times G_d$. For any word or permutation $v \in [m]^N$, define the i^{th} *block of* v to be the subword $v^{(i)} = v_{1+N_i}v_{2+N_i}\cdots v_{n_i+N_i}$. Note that v is the concatenation of $v^{(1)}, v^{(2)}, \ldots, v^{(d)}$, and $\mathrm{stdz}(v^{(i)})$ is a permutation in S_{n_i} for $1 \leq i \leq d$.

The following proposition is readily proved from (4).

Proposition 8. *With the above notation,* $\mathrm{cyc}_G = \prod_{i=1}^d \mathrm{cyc}_{G_i}$ *holds by a bijective proof. The weight-preserving bijection sends* $[w]_G \in [m]^N/G$ *to*

$$([w^{(1)}]_{G_1}, [w^{(2)}]_{G_2}, \dots, [w^{(d)}]_{G_d}) \in [m]^{n_1}/G_1 \times [m]^{n_2}/G_2 \times \cdots \times [m]^{n_d}/G_d.$$

To prove Theorem 10, we need a bijective version of the multiplication rule (6) for fundamental quasisymmetric polynomials. The following observation plays a key role in the proof: if $v \in [m]^N$ and $s = \mathrm{stdz}(v) \in S_N$, then $\mathrm{stdz}(v^{(i)}) = \mathrm{stdz}(s^{(i)})$ for $1 \le i \le d$. In other words, standardizing the i^{th} block of consecutive symbols in v gives the same result as first standardizing all of v, and then standardizing the i^{th} block of the resulting permutation. For example, suppose $N = 8$, $n_1 = 5$, $n_2 = 3$, and $v = 34132414$. Then $s = 46153728$, $v^{(1)} = 34132$, $s^{(1)} = 46153$, $\mathrm{stdz}(v^{(1)}) = 35142 = \mathrm{stdz}(s^{(1)})$, $v^{(2)} = 414$, $s^{(2)} = 728$, and $\mathrm{stdz}(v^{(2)}) = 213 = \mathrm{stdz}(s^{(2)})$.

The following proposition is readily proved from (8).

Proposition 9. *For all* $t^{(1)} \in S_{n_1}$, $t^{(2)} \in S_{n_2}$, $\dots$, $t^{(d)} \in S_{n_d}$,

$$\prod_{i=1}^{d} F_{n_i,\mathrm{IDes}(t^{(i)})}(x_1, \dots, x_m) = \sum_{\substack{s \in S_N:\ \mathrm{stdz}(s^{(i)}) = t^{(i)} \\ \text{for } 1 \le i \le d}} F_{N,\mathrm{IDes}(s)}(x_1, \dots, x_m) \tag{13}$$

holds by a bijective proof. The bijection sends $(w^{(1)}, w^{(2)}, \dots, w^{(d)}) \in U(t^{(1)}) \times \cdots \times U(t^{(d)})$ *to the concatenation* $w^{(1)}w^{(2)} \cdots w^{(d)}$.

We remark that (13) can be deduced algebraically from (6) by letting $t^{(i)} = [s^{(i)}]^{-1}$, but we need the bijective version.

Theorem 10. *For* $1 \le i \le d$, *suppose* G_i *is a subgroup of* S_{n_i} *and* $\mathcal{C}_i$ *is a subset of* S_{n_i} *such that*

$$\mathrm{cyc}_{G_i}(x_1, \dots, x_m) = \sum_{t^{(i)} \in \mathcal{C}_i} F_{n_i,\mathrm{IDes}(t^{(i)})}(x_1, \dots, x_m). \tag{14}$$

Let $\mathcal{C} = \{s \in S_N : \mathrm{stdz}(s^{(i)}) \in \mathcal{C}_i \text{ for } 1 \le i \le d\}$.
(a) The subgroup $G = G_1' \times \cdots \times G_d'$ *in* S_N *satisfies*

$$\mathrm{cyc}_G(x_1, \dots, x_m) = \sum_{s \in \mathcal{C}} F_{N,\mathrm{IDes}(s)}(x_1, \dots, x_m). \tag{15}$$

(b) Bijective proofs of (14) *for* $1 \le i \le d$ *combine to give a bijective proof of* (15). *(c) If the standardization approach succeeds for each* G_i, *then the standardization approach succeeds for* G. *(d) If the standardization approach fully succeeds for each* G_i, *then the standardization approach fully succeeds for* G.

Proof. Parts (a) and (b) follow easily from Propositions 8 and 9.

For (c), assume that the standardization approach succeeds for each G_i using the collection $\mathcal{C}_i$. We prove that the standardization approach also succeeds for

G using the collection $\mathcal{C}$. Given any orbit $[z]_G \in [m]^N$, we must show there exists a unique w in this orbit such that $\mathrm{stdz}(w) \in \mathcal{C}$. Now, for any word $w \in [m]^N$, the definition of G shows that the concatenation map restricts to a bijection from $[z^{(1)}]_{G_1} \times \cdots \times [z^{(d)}]_{G_d}$ to $[z]_G$. So for every $w \in [z]_G$, there exist unique $u^{(i)} \in [z^{(i)}]_{G_i}$ such that w is the concatenation of $u^{(1)}, \ldots, u^{(d)}$. Now, $s = \mathrm{stdz}(w) \in \mathcal{C}$ iff $\mathrm{stdz}(s^{(i)}) \in \mathcal{C}_i$ for all i iff $\mathrm{stdz}(w^{(i)}) \in \mathcal{C}_i$ for all i iff $\mathrm{stdz}(u^{(i)}) \in \mathcal{C}_i$ for all i. By assumption, for each i, there exists a unique $v^{(i)} \in [z^{(i)}]_{G_i}$ that standardizes to something in $\mathcal{C}_i$. We conclude that there exists a unique $v \in [z]_G$ that standardizes to something in $\mathcal{C}$, namely the concatenation of $v^{(1)}, \ldots, v^{(d)}$. So (10) holds for G and $\mathcal{C}$.

For (d), assume that the standardization approach fully succeeds for each G_i. For $1 \leq i \leq d$, let $\mathcal{D}_i$ be a collection of subsets of $[n_i - 1]$ such that (14) holds with $\mathcal{C}_i = \{s \in S_{n_i} : \mathrm{Des}(s) \in \mathcal{D}_i\}$. Define $X_i = \{N_i + 1, N_i + 2, \ldots, N_i + n_i - 1\}$ for $1 \leq i \leq d$, and let $\mathcal{D}'_i$ be the collection of subsets of X_i obtained by adding N_i to every element of every subset in $\mathcal{D}_i$. Define

$$\mathcal{D} = \{D \subseteq [N-1] : D \cap X_i \in \mathcal{D}'_i \text{ for } 1 \leq i \leq d\}. \tag{16}$$

We must prove $\mathcal{C} = \{s \in S_N : \mathrm{Des}(s) \in \mathcal{D}\}$. Given any $s \in S_N$ and i between 1 and d, one readily checks that $\mathrm{Des}(\mathrm{stdz}(s^{(i)})) \in \mathcal{D}_i$ iff $\mathrm{Des}(s) \cap X_i \in \mathcal{D}'_i$, since standardizing a word with no repeated letters does not create or destroy descents. So $s \in \mathcal{C}$ iff $\mathrm{stdz}(s^{(i)}) \in \mathcal{C}_i$ for all $i \in [d]$ iff $\mathrm{Des}(\mathrm{stdz}(s^{(i)})) \in \mathcal{D}_i$ for all $i \in [d]$ iff $\mathrm{Des}(s) \cap X_i \in \mathcal{D}'_i$ for all $i \in [d]$ iff $\mathrm{Des}(s) \in \mathcal{D}$. $\square$

4.2. Solution for Young Subgroups and Embedded Subgroups

In this subsection, we give two applications of Theorem 10. A *Young subgroup* of S_N is a subgroup of the form $S'_{n_1} \times S'_{n_2} \times \cdots \times S'_{n_d}$, where we use the notation from the beginning of §4.1. Since $\mathrm{cyc}_{S_{n_i}} = F_{n_i,\emptyset}$ by (11), Theorem 10 yields the following result (take $\mathcal{C}_i = \{\mathrm{id}_{n_i}\}$ and $\mathcal{D}_i = \{\emptyset\}$ for all i).

Corollary 11. *Given positive integers $n_1, n_2, \ldots, n_d$ with sum N, let G be the Young subgroup $S'_{n_1} \times S'_{n_2} \cdots \times S'_{n_d}$ of S_N. The standardization approach fully succeeds for G, giving a bijective proof that*

$$\mathrm{cyc}_G(x_1, \ldots, x_m) = \sum_{\substack{s \in S_N: \\ \mathrm{Des}(s) \subseteq \{n_1, n_1+n_2, \ldots, n_1+\cdots+n_{d-1}\}}} F_{N,\mathrm{IDes}(s)}(x_1, \ldots, x_m).$$

Remark 12. We know $\mathrm{cyc}_{S_{n_i}} = F_{n_i,\emptyset} = h_{n_i}$, the complete homogeneous symmetric polynomial. Therefore, for $G = S'_{n_1} \times \cdots \times S'_{n_d}$, $\mathrm{cyc}_G = h_{n_1} h_{n_2} \cdots h_{n_d} = h_\mu$ where $\mu = \mathrm{sort}(n_1, n_2, \ldots, n_d)$. Thus, Corollary 11 provides the F-expansions of the symmetric polynomials h_μ. We obtain another formula for these expansions by combining (7) with the well-known identity $h_\mu = \sum_\lambda K_{\lambda,\mu} s_\lambda$, where the *Kostka numbers* $K_{\lambda,\mu}$ count semistandard tableaux of shape λ and content μ. The Schur expansion of h_μ can be proved combinatorially using the tableau insertion algorithm (see, for instance, [20, §9.12]) or by representation-theoretical arguments (see [28, §2.11]). In §5, we show how the standardization approach leads to a bijective proof of this Schur expansion.

For our second application of Theorem 10, suppose G is a subgroup of S_n and $N > n$. S_N contains an isomorphic copy of G, namely $G^* = G \times S'_1 \times \cdots \times S'_1$, where there are $N-n$ factors isomorphic to $S_1 = \{(1)\}$. So $\mathrm{cyc}_{G^*} = \mathrm{cyc}_G \cdot F_{1,\emptyset}^{N-n}$, where $F_{1,\emptyset} = x_1 + x_2 + \cdots + x_m$. Note that every one-letter word standardizes to 1. Combining this remark with Theorem 10 immediately yields the following result.

Corollary 13. *Suppose G is a subgroup of S_n and $\mathcal{C}$ is a subset of S_n such that $\mathrm{cyc}_G = \sum_{s\in\mathcal{C}} F_{n,\mathrm{IDes}(s)}$. For $N \geq n$, let G^* be G viewed as a subgroup of S_N, and let $\mathcal{C}^* = \{s \in S_N : \mathrm{stdz}(s_1 \cdots s_n) \in \mathcal{C}\}$. Then $\mathrm{cyc}_{G^*} = \sum_{s\in\mathcal{C}^*} F_{N,\mathrm{IDes}(s)}$. Moreover, if the standardization approach succeeds (resp. fully succeeds) for G and $\mathcal{C}$, then the standardization approach succeeds (resp. fully succeeds) for G^* and $\mathcal{C}^*$.*

4.3. Solution for Conjugate Subgroups

Given two subgroups G and H of S_n, recall that H is *conjugate* to G iff there exists $s \in S_n$ with $H = sGs^{-1}$. The next theorem follows easily from the fact that conjugate permutations have the same cycle type.

Theorem 14. *For all subgroups G and H of S_n, if H is conjugate to G, then $\mathrm{cyc}_H = \mathrm{cyc}_G$. Moreover, any bijective proof of the F-expansion or Schur expansion of cyc_G induces a bijective proof of the same expansion for cyc_H.*

For example, suppose $H = sGs^{-1}$ and the bijection f for cyc_G comes from the standardization approach using a collection $\mathcal{C}$ satisfying (10). The corresponding bijection for cyc_H starts with an orbit $[w]_H \in [m]^n/H$, maps this orbit to $[w \circ s]_G$, finds the unique word $v \in [w \circ s]_G \cap U(\mathcal{C})$, and returns the answer $(\mathrm{stdz}(v), \mathrm{sort}(v))$.

4.4. Reduction to Compressed Words

If G is a subgroup of S_n for a small value of n, then it is possible to check condition (10) of Theorem 4 by an exhaustive computer search. Our next result shows that such a search only needs to check some of the orbits in $[m]^n/G$.

Define a word $w \in [m]^n$ to be *compressed* iff for all $j < k$ in $[m]$, if k appears in w then j appears in w. Given any word $z \in [m]^n$, the *compression of* z is the word $c(z) \in [n]^n$ defined as follows. Let the distinct letters appearing in z be $i_1 < i_2 < \cdots < i_s$, where $s \leq n$ since z has length n. Form $c(z)$ by replacing each occurrence of i_j in z by j, for $1 \leq j \leq s$. For example, $c(47726274) = 24413142$. One easily checks that $c(z)$ is a compressed word in $[n]^n$ such that $\mathrm{stdz}(c(z)) = \mathrm{stdz}(z)$. Moreover, for all $g \in S_n$ and all $z, z' \in [m]^n$, $g\star z = z'$ implies $g\star c(z) = c(z')$. So, for any subgroup G of S_n and any $z \in [m]^n$, $[c(z)]_G = \{c(z') : z' \in [z]_G\}$.

Theorem 15. *Fix a subgroup G of S_n and a subset $\mathcal{C}$ of S_n. Suppose that for every compressed word $w \in [n]^n$, $U(\mathcal{C})$ intersects the orbit $[w]_G$ in exactly one point. Then (10) holds for G, $\mathcal{C}$, and any m.*

Proof. Assume (10) fails for some orbit $[z]_G \in [m]^n/G$; we show that (10) also fails for $[w]_G$, where w is the compressed word $c(z)$. On one hand, suppose $U(\mathcal{C}) \cap [z]_G = \emptyset$, which means $\mathrm{stdz}(g \star z) \notin \mathcal{C}$ for all $g \in G$. Then for all $g \in G$, $\mathrm{stdz}(g \star w) = \mathrm{stdz}(g \star c(z)) = \mathrm{stdz}(c(g \star z)) = \mathrm{stdz}(g \star z) \notin \mathcal{C}$, so that $U(\mathcal{C}) \cap [w]_G = \emptyset$. On the other hand, suppose $z_1 = g_1 \star z$ and $z_2 = g_2 \star z$ are two distinct words in $[z]_G$ such that $\mathrm{stdz}(z_1), \mathrm{stdz}(z_2) \in \mathcal{C}$. Then $w_1 = c(z_1) = g_1 \star w$ and $w_2 = c(z_2) = g_2 \star w$ are two distinct words in $[w]_G$ such that $\mathrm{stdz}(w_1), \mathrm{stdz}(w_2) \in \mathcal{C}$. □

4.5. Analysis of $\langle(1, 2, 3, 4)\rangle$

Theorem 15, Corollary 13, and routine computer calculations yield the following result.

Proposition 16. *For* $n \geq 4$*, let* $G = \langle(1, 2, 3, 4)\rangle \leq S_n$*. Define*

$$\mathcal{C}_1 = \{1234, 1324, 1432, 2431, 3421, 4231\} \text{ and}$$
$$\mathcal{C}_2 = \{1234, 1324, 3214, 4213, 4231, 4312\}.$$

The standardization strategy bijectively proves

$$\mathrm{cyc}_G = \sum_{s \in S_n :\, \mathrm{stdz}(s_1 s_2 s_3 s_4) \in \mathcal{C}_1} F_{n,\mathrm{IDes}(s)} = \sum_{s \in S_n :\, \mathrm{stdz}(s_1 s_2 s_3 s_4) \in \mathcal{C}_2} F_{n,\mathrm{IDes}(s)}.$$

When $n = 4$*,* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *are the only choices of* $\mathcal{C}$ *satisfying* (10)*.*

One readily checks that $\mathcal{C}_1$ and $\mathcal{C}_2$ do not have the form (2) for any choice of $\mathcal{D}$. Thus *the standardization approach succeeds, but does not fully succeed, for* $G = \langle(1, 2, 3, 4)\rangle$. Nevertheless, there is an algebraic formula for cyc_G involving a set $\mathcal{C}$ satisfying (2). Using Proposition 16 with $n = 4$, one verifies by direct calculation that $\mathrm{cyc}_G = \sum_{s \in S_4 : \mathrm{Des}(s) \in \{\emptyset, \{1,3\}\}} F_{4,\mathrm{IDes}(s)}$. Corollary 13 now applies to give the following algebraic result.

Corollary 17. *Let* G *be* $\langle(1, 2, 3, 4)\rangle$ *viewed as a subgroup of* S_n *for* $n \geq 4$*. Then*

$$\mathrm{cyc}_G = \sum_{\substack{s \in S_n : \\ \mathrm{Des}(s_1 s_2 s_3 s_4) \in \{\emptyset, \{1,3\}\}}} F_{n,\mathrm{IDes}(s)}.$$

4.6. Analysis of the Dihedral Group of Order 8

Let $G = \langle(1, 2, 3, 4)\rangle = \{1234, 2341, 3412, 4123\}$, and let D be the eight-element dihedral group $D = G \cup \{4321, 3214, 2143, 1432\}$. For each $w \in [m]^4$, $[w]_D = [w]_G \cup [4321 \star w]_G$, where $4321 \star w$ is the reversal of the word w. This means that each D-orbit of w either coincides with the G-orbit of w or is the union of two G-orbits of w. The same search process used to prove Proposition 16 establishes the following result.

Proposition 18. *For $n \geq 4$, let D be the 8-element dihedral group viewed as a subgroup of S_n. Define $\mathcal{C}_1 = \{1234, 1324, 2431\}$, $\mathcal{C}_2 = \{1234, 1324, 4213\}$, $\mathcal{C}_3 = \{1234, 3142, 2431\}$, and $\mathcal{C}_4 = \{1234, 3142, 4213\}$. For $1 \leq k \leq 4$, the standardization strategy bijectively proves*

$$\mathrm{cyc}_D = \sum_{s \in S_n :\, \mathrm{stdz}(s_1 s_2 s_3 s_4) \in \mathcal{C}_k} F_{n,\mathrm{IDes}(s)}.$$

For $n = 4$, the four sets $\mathcal{C}_k$ are the only choices of $\mathcal{C}$ satisfying (10).

Here too, none of the sets $\mathcal{C}_k$ have the form (2) for any choice of $\mathcal{D}$. So *the standardization approach succeeds, but does not fully succeed, for the group D*. When $n = 4$, $\mathrm{cyc}_D = F_{4,\emptyset} + F_{4,\{2\}} + F_{4,\{1,3\}}$. Even algebraically, this F-expansion cannot be written in the form $\sum_{s \in S_4 : \mathrm{Des}(s) \in \mathcal{D}} F_{n,\mathrm{IDes}(s)}$ for any $\mathcal{D}$.

5. Schur Expansions

When the standardization approach fully succeeds for a given group G, we can convert the F-expansion of cyc_G into a Schur expansion, as shown in the next theorem.

Theorem 19. *Suppose G is a subgroup of S_n and $\mathcal{D}$ is a collection of subsets of $[n-1]$ such that*

$$\mathrm{cyc}_G(x_1, \ldots, x_m) = \sum_{s \in S_n :\, \mathrm{Des}(s) \in \mathcal{D}} F_{n,\mathrm{IDes}(s)}(x_1, \ldots, x_m). \tag{17}$$

Then

$$\mathrm{cyc}_G(x_1, \ldots, x_m) = \sum_{\lambda \in \mathrm{Par}(n)} |\{Q \in \mathrm{SYT}(\lambda) : \mathrm{Des}(Q) \in \mathcal{D}\}| s_\lambda(x_1, \ldots, x_m). \tag{18}$$

Moreover, any bijective proof of (17) *induces a bijective proof of* (18).

Proof. Given (s, v) with $s \in S_n$, $\mathrm{Des}(s) \in \mathcal{D}$, and $v \in W_{n,\mathrm{IDes}(s)}$, use the Robinson–Schensted correspondence to convert s to a pair (P, Q) of standard tableaux of the same shape λ. Map (s, v) to the triple (λ, Q, T), where T is the semistandard tableau of weight $\mathrm{wt}(v)$ obtained from P by replacing i in P by v_i for $1 \leq i \leq n$. One readily checks that this defines a weight-preserving bijection proving that the right sides of (17) and (18) are equal. □

More generally, suppose we replace condition (2) by the following weaker condition: for some collection $\mathcal{Q}$ of standard tableaux with n cells, $\mathcal{C}$ is the set of $s \in S_n$ such that the RSK recording tableau $Q(s)$ lies in $\mathcal{Q}$. By the same bijective proof just given, we see that

$$\sum_{s \in \mathcal{C}} F_{n,\mathrm{IDes}(s)} = \sum_{Q \in \mathcal{Q}} s_{\mathrm{shape}(Q)}. \tag{19}$$

This offers us more chances to obtain Schur expansions of cycle index polynomials by using the standardization approach. Unfortunately, one can check that the sets $\mathcal{C}$ appearing in Propositions 16 and 18 do not satisfy the weaker condition stated here. So this approach by itself is not powerful enough to provide bijective proofs of the Schur expansions for every subgroup G.

Remark 20. We can use representation theory to see that for every subgroup G of S_n, there is always an *algebraic* formula of the form (19) for cyc_G, although we are not guaranteed a *combinatorial* or *bijective* proof of this formula. It suffices to note that the left regular S_n-module $\mathbb{C}[S_n]$ has Frobenius characteristic $\sum_\lambda |\mathrm{SYT}(\lambda)| s_\lambda$, and the module $\mathbb{C}[S_n/G]$ with Frobenius characteristic cyc_G is a quotient module of $\mathbb{C}[S_n]$. We leave it as an open question to find explicit descriptions of the collections $\mathcal{Q}$ and bijective proofs of these Schur expansions valid for general subgroups G.

6. Analysis of the Cyclic Groups $\langle(1, 2, \ldots, p)\rangle$ for Prime p

This section studies the cycle index polynomials cyc_G when p is an odd prime and G is the cyclic subgroup $\langle(1, 2, \ldots, p)\rangle$ in S_p. We prove that *the standardization approach fully succeeds for these groups.* The first key concept is the notion of a circular descent set, which reduces the problem to finding perfect matchings in a certain graph.

6.1. Circular Descent Sets

Given any word $w \in [m]^n$, define the *circular descent set* $\mathrm{CDes}(w) = \{i \in [n] : w_i > w_{i+1}\}$ where subscripts are reduced modulo n. In particular, $n \in \mathrm{CDes}(w)$ iff $w_n > w_1$. For example, $\mathrm{CDes}(144323) = \{3, 4, 6\}$ and $\mathrm{CDes}(362451) = \{2, 5\}$. For every word w of length n, $\mathrm{Des}(w) = \mathrm{CDes}(w) \setminus \{n\}$.

Lemma 21. *For all words $w \in [m]^n$, $\mathrm{Des}(w) = \mathrm{Des}(\mathrm{stdz}(w))$.*

Proof. Suppose $w = w_1w_2 \cdots w_n \in [m]^n$ standardizes to an element $s = \mathrm{stdz}(w)$ in S_n given by $s = s_1s_2 \cdots s_n$; we prove $\mathrm{Des}(w) = \mathrm{Des}(s)$. Fix i in the range $1 \leq i < n$. If $i \in \mathrm{Des}(w)$, then $w_i > w_{i+1}$, so standardizing relabels w_i with a larger symbol than the symbol used to relabel w_{i+1}. Thus $s_i > s_{i+1}$, which means $i \in \mathrm{Des}(s)$. If $i \notin \mathrm{Des}(w)$, then $w_i \leq w_{i+1}$. In the case $w_i < w_{i+1}$, then $s_i < s_{i+1}$ follows as before. In the case $w_i = w_{i+1}$, then $s_{i+1} = s_i + 1$ since standardization renumbers equal values in w from left to right (and $i < n$ here). So $s_i < s_{i+1}$ holds, and hence $i \notin \mathrm{Des}(s)$ in both cases. □

The next theorem shows how circular descent sets can help us find collections $\mathcal{C}$ and $\mathcal{D}$ satisfying (1) and (2). From now on, we identify a subset S of $[n]$ with the bit string $b \in \{0,1\}^n$ such that $i \in S$ iff $b_i = 1$. In this notation, we obtain $\mathrm{Des}(w)$ from $\mathrm{CDes}(w)$ by deleting the last bit. We also write $\{0,1\}^p/G$ for the set of orbits of binary words under the cyclic shifting action of $G = \langle(1, 2, \ldots, p)\rangle$.

Theorem 22. *Let p be an odd prime, and let $G = \langle(1, 2, \ldots, p)\rangle$ in S_p. Suppose $\mathcal{D} \subseteq \{0, 1\}^{p-1}$ satisfies this condition:*

$$\textit{For all } b \in \{0,1\}^p \textit{ there is a unique } d \in \mathcal{D} \textit{ with } d0 \in [b]_G \textit{ or } d1 \in [b]_G. \tag{20}$$

Then for all positive integers m,

$$\mathrm{cyc}_G(x_1, \ldots, x_m) = \sum_{s\in S_n\colon \mathrm{Des}(s)\in\mathcal{D}} F_{n,\mathrm{IDes}(s)}(x_1, \ldots, x_m),$$

so the standardization approach fully succeeds for G.

Proof. It suffices to check (10), taking $\mathcal{C} = \{s \in S_n : \mathrm{Des}(s) \in \mathcal{D}\}$. Fix $z \in [m]^p$, and consider two cases. Case 1: z is a constant word, meaning that all letters of z are equal. Then $[z]_G = \{z\}$ and $\mathrm{stdz}(z) = \mathrm{id}_p = 12\cdots p$. In (20), take $b = 00\cdots 0 \in \{0, 1\}^p$, so that $[b]_G = \{b\}$. We see that $\mathcal{D}$ must contain the bit string $d = 0\cdots 0 \in \{0, 1\}^{p-1}$, which encodes the empty set. Since $\emptyset$ belongs to $\mathcal{D}$, id_p belongs to $\mathcal{C}$, so $[z]_G \cap U(\mathcal{C})$ consists of the single word z.

Case 2: z is not a constant word, so the orbit $[z]_G$ does not consist of z alone. Since p is prime, this forces $[z]_G$ to contain p distinct words $z^{(1)}, \ldots, z^{(p)}$, which are the p different cyclic shifts of z. Let $b = \mathrm{CDes}(z)$ viewed as a bit string in $\{0, 1\}^p$, and let $b^{(i)} = \mathrm{CDes}(z^{(i)})$ for $1 \le i \le p$. Since cyclically shifting z also cyclically shifts the locations of the circular descents in z, we see that $[b]_G = \{b^{(1)}, b^{(2)}, \ldots, b^{(p)}\}$. We claim b cannot be a constant bit string. For if $b_i = 1$ for all i, we would have $z_1 > z_2 > \cdots > z_p > z_1$, which is impossible. And if $b_i = 0$ for all i, we would have $z_1 \le z_2 \le \cdots \le z_p \le z_1$, which would force z to be a constant word. Thus the p words $b^{(1)}, \ldots, b^{(p)}$ are all distinct.

To finish, note that $z^{(i)}$ is in $U(\mathcal{C})$ iff $\mathrm{stdz}(z^{(i)})$ is in $\mathcal{C}$ iff $\mathrm{Des}(\mathrm{stdz}(z^{(i)}))$ is in $\mathcal{D}$ iff $\mathrm{Des}(z^{(i)})$ is in $\mathcal{D}$ iff $b^{(i)}$ with its last bit deleted is in $\mathcal{D}$. By the assumed condition (20), the final condition on $b^{(i)}$ holds for exactly one i between 1 and p. Therefore, $z^{(i)} \in U(\mathcal{C})$ holds for exactly one i between 1 and p. This proves that every G-orbit in $[m]^n$ intersects $U(\mathcal{C})$ in exactly one point, as needed. $\square$

6.2. Reduction to Perfect Matching Problem

To find formulas for $\mathrm{cyc}_G(x_1, \ldots, x_m)$ when $G = \langle(1, 2, \ldots, p)\rangle$, we must still find collections $\mathcal{D} \subseteq \{0, 1\}^{p-1}$ satisfying condition (20). This condition can be reformulated in graph-theoretic terms, as follows. Let $\mathcal{G}_p$ be the graph with vertex set $V(\mathcal{G}_p) = \{0, 1\}^p/G$ and edge set

$$E(\mathcal{G}_p) = \{\{[d0]_G, [d1]_G\} : d \in \{0, 1\}^{p-1}\}.$$

We obtain $\mathcal{G}_p$ from the hypercube $\{0, 1\}^p$ by identifying all vertices that differ by a cyclic shift. Note that $\mathcal{G}_p$ can have multiple edges between the same two vertices, but there are no loop edges since $[d0]_G \ne [d1]_G$ for all d. A subset $\mathcal{D}$ of $\{0, 1\}^{p-1}$ satisfies (20) iff the corresponding set of edges $M = \{\{[d0]_G, [d1]_G\} : d \in \mathcal{D}\}$ is a *perfect matching* of $\mathcal{G}_p$. Recall this means that every vertex of $\mathcal{G}_p$ is the endpoint of exactly one edge in M.

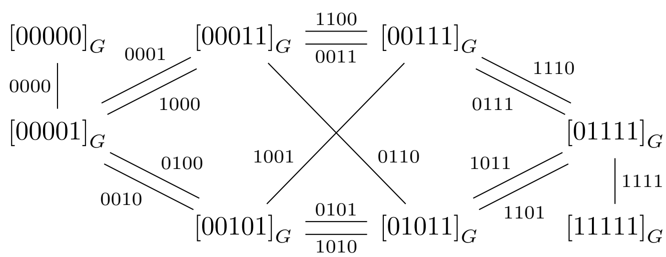

Figure 1: The graph $\mathcal{G}_5$.

As an example, Figure 1 shows the graph $\mathcal{G}_5$. We see by inspection of the graph that $\mathcal{G}_5$ has five perfect matchings, corresponding to the following subsets of $\{0,1\}^4$:

$$\begin{array}{lll} \mathcal{D}_1 & = \{0000, 0110, 1001, 1111\} & = \{\emptyset, \{2,3\}, \{1,4\}, \{1,2,3,4\}\}; \\ \mathcal{D}_2 & = \{0000, 0011, 0101, 1111\} & = \{\emptyset, \{3,4\}, \{2,4\}, \{1,2,3,4\}\}; \\ \mathcal{D}_3 & = \{0000, 1100, 0101, 1111\} & = \{\emptyset, \{1,2\}, \{2,4\}, \{1,2,3,4\}\}; \\ \mathcal{D}_4 & = \{0000, 0011, 1010, 1111\} & = \{\emptyset, \{3,4\}, \{1,3\}, \{1,2,3,4\}\}; \\ \mathcal{D}_5 & = \{0000, 1100, 1010, 1111\} & = \{\emptyset, \{1,2\}, \{1,3\}, \{1,2,3,4\}\}. \end{array} \tag{21}$$

Applying Theorem 22 and Corollary 13, we deduce the following result.

Proposition 23. *Let $G = \langle(1,2,3,4,5)\rangle$ viewed as a subgroup of S_n. For each collection $\mathcal{D}_k$ shown in* (21)*,*

$$\mathrm{cyc}_G(x_1,\ldots,x_m) = \sum_{\substack{s \in S_n: \\ \mathrm{Des}(s_1\cdots s_5) \in \mathcal{D}_k}} F_{n,\mathrm{IDes}(s)}(x_1,\ldots,x_m).$$

6.3. Construction of Perfect Matchings for Prime p

Kramer, Lastaria, and Salvi proved that for all odd n, the graph $\mathcal{G}_n$ has at least one perfect matching [18]. (Those authors begin with the n-dimensional binary de Bruijn graph, collapse vertices that differ by a cyclic shift, and ignore edge directions. One readily checks that this produces the graph $\mathcal{G}_n$, disregarding multiple edges.) Their proof proceeds algorithmically, showing that any non-perfect matching M of $\mathcal{G}_n$ can be enlarged by finding an M-augmenting path.

In this section, we explicitly exhibit a particular perfect matching for $\mathcal{G}_p$ for every odd prime p. This provides a formula for $\mathrm{cyc}_{\langle(1,2,\ldots,p)\rangle}$ like that given in Proposition 23. In general, there are many such formulas, one for each perfect matching of $\mathcal{G}_p$. For instance, $\mathcal{G}_7$ has 6285 perfect matchings. For simplicity, in what follows we identify all edges in $\mathcal{G}_p$ with the same endpoints, thus viewing $\mathcal{G}_p$ as a simple graph. This affects the number of perfect matchings but not the existence of a perfect matching.

There are three key ingredients in our construction. The first is to find a perfect matching on a particular path in $\mathcal{G}_p$ connecting the vertices $[00\cdots 0]_G$

and $[11\cdots1]_G$. The second is to encode the remaining vertices of $\mathcal{G}_p$ as circular compositions, as is done in [18]. The third is to group the vertices of $\mathcal{G}_p$ into disjoint binary hypercubes of various dimensions. A perfect matching can then be chosen independently on each such hypercube.

We begin by matching the vertices on the path from $[00\cdots0]_G$ to $[11\cdots1]_G$. For $0 \le a \le p$, let v^a be the word consisting of a zeros followed by $p-a$ ones. For a odd, match $[v^a]_G$ with $[v^{a-1}]_G$; equivalently, for a even, match $[v^a]_G$ with $[v^{a+1}]_G$. Define

$$\mathcal{G}_p' = \mathcal{G}_p \setminus \{[v^0]_G, [v^1]_G, \ldots, [v^p]_G\}.$$

For the second step, we encode the vertices of $\mathcal{G}_p'$ as circular compositions. Let C_p denote the set of integer compositions of p. That is,

$$C_p = \{(c_1, c_2, \ldots, c_\ell) :\ \ell > 0,\ c_1, \ldots, c_\ell > 0,\ c_1 + \cdots + c_\ell = p\}.$$

Let C_p^* be the set of equivalence classes of C_p under cyclic rotation of the parts. For example, using square brackets to denote equivalence classes, we have

$$\begin{aligned} C_5^* = \{&\{11111\}, \{1112, 1121, 1211, 2111\}, \{122, 212, 122\}, \{113, 131, 311\},\\ &\{23, 32\}, \{14, 41\}, \{5\}\} = \{[11111], [1112], [122], [113], [23], [14], [5]\}. \end{aligned}$$

Because p is prime, all ℓ cyclic rotations of any composition $c = (c_1, c_2, \ldots, c_\ell)$ of p must be distinct.

We define a map $\phi : V(\mathcal{G}_p') \to C_p^*$ as follows. Given $v = v_1 v_2 \cdots v_p \in \{0,1\}^p$, let $i_1 < i_2 < \cdots < i_k$ be the indices for which $v_j = 1$. Note that $k \ge 1$ since $[00\cdots0]_G \notin V(\mathcal{G}_p')$. Define

$$\phi([v]_G) = [(i_2 - i_1, i_3 - i_2, \ldots, i_k - i_{k-1}, (p + i_1) - i_k)].$$

We can compute $\phi([v]_G)$ by replacing each string of b 0s followed cyclically by a 1 with the part $b+1$. For example, $\phi([0011000]_G) = [16]$ and $\phi([1000101]_G) = [421]$. One readily checks that ϕ *is well-defined and injective.*

Let $\mathcal{G}_p^*$ be the graph with vertex set $\{\phi([v]_G) : v \in \mathcal{G}_p'\}$ and edge set $\{\{\phi([v]_G), \phi([w]_G)\} : \{[v]_G, [w]_G\} \in E(\mathcal{G}_p')\}$. One readily checks that $\{[c], [d]\}$ is an edge in $\mathcal{G}_p^*$ iff d is obtained from c by combining two consecutive parts of c (reading parts cyclically) or splitting one part of c into two parts. Moreover, ϕ is a graph isomorphism between $\mathcal{G}_p'$ and $\mathcal{G}_p^*$. For example, Figure 2 illustrates the graphs $\mathcal{G}_7'$ and $\mathcal{G}_7^*$.

Theorem 24. *For every odd prime p, $\mathcal{G}_p^*$ has a perfect matching.*

Proof. Let $c = (c_1, c_2, \ldots, c_\ell)$ be a circular composition with $[c] \in V(\mathcal{G}_p^*)$, so c has at least two parts greater than 1. We match $[c]$ to another vertex by applying one of the following moves. A *split* move replaces a certain part $M > 1$ in $[c]$ by successive parts 1 and $M-1$. A *combine* move replaces successive parts 1 and M in $[c]$ by $1 + M$. We split if the number of parts of size 1 immediately preceding the part M (taking wraparound into account) is even and combine

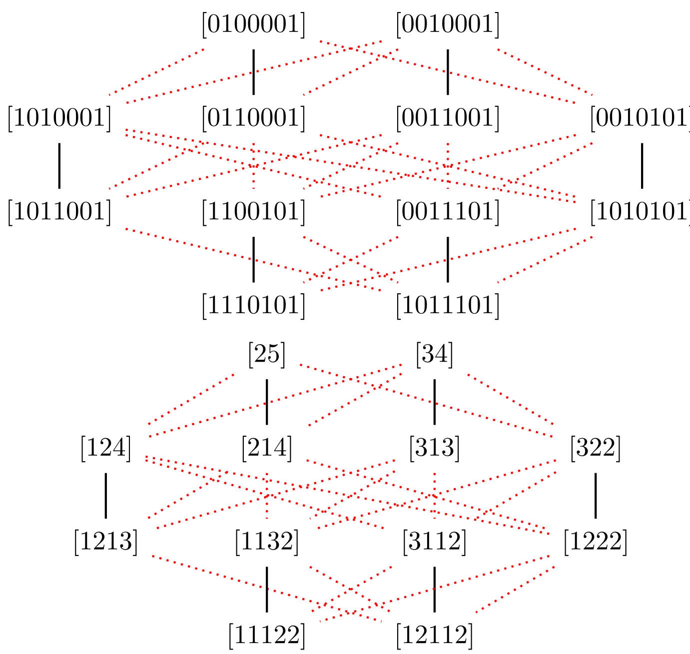

Figure 2: The graphs $\mathcal{G}'_7$ and $\mathcal{G}^*_7$. The solid edges form the perfect matching described in the proof of Theorem 24.

if this number is odd. The tricky point is determining which part of $[c]$ should play the role of M.

Our approach is to identify a hypercube $\mathrm{HC}([c])$ in $\mathcal{G}^*_p$ having $[c]$ as a vertex. We partially order the set of circular compositions by setting $[c_1,\ldots,c_\ell] \prec [d_1,\ldots,d_{\ell'}]$ if and only if $\ell > \ell'$. Let $M = \max(c_1, c_2, \ldots, c_\ell)$, and let $A = \{a_1 < a_2 < \cdots < a_p\}$ be the indices for which $c_{a_i} = M$. Note that $M > 1$ since $[11\cdots 1] \notin V(\mathcal{G}^*_p)$. For each $a_i \in A$, let y_i be the number of parts of size 1 immediately preceding c_{a_i} (taking wraparound into account). We now consider cases that correspond to whether $[c]$ is the minimum vertex of $\mathrm{HC}([c])$ with respect to the partial order $\prec$.

Case I: At least one y_i is odd.
Let $A_{\text{odd}} = \{a_i \in A : y_i \text{ is odd}\}$ and $k = |A_{\text{odd}}|$. For each $y_i \in A_{\text{odd}}$, we could choose to replace the cyclically consecutive parts $c_{a_i-1} = 1$ and $c_{a_i} = M$ by a single part $1 + M$. Treating all of these k choices independently, we obtain the 2^k vertices of a k-dimensional hypercube denoted $\mathrm{HC}([c])$, which appears as a subgraph of $\mathcal{G}^*_p$. Note that all circular compositions arising from c in this way still have at least two parts greater than 1.

Here are three examples of this construction. (1) If $c = 315$, then $\mathrm{HC}([c])$ is the 1-dimensional hypercube with vertices $[315]$ and $[36]$. (2) If $c = 144$, then $\mathrm{HC}([c])$ is the 1-dimensional hypercube with vertices $[144]$ and $[54]$. (3) If $c = 143111424$, then $M = 4$, $A = \{2, 7, 9\}$, $y_1 = 1$, $y_2 = 3$, and $y_3 = 0$, so

$A_{\mathrm{odd}} = \{2, 7\}$. We could combine the parts $c_1 = 1$ and $c_2 = 4$ into a single part 5. We could also combine the parts $c_6 = 1$ and $c_7 = 4$ into a single part 5. Thus, $\mathrm{HC}([c])$ is the 2-dimensional hypercube with vertices [143111424], [14311524], [53111424], and [5311524].

There are many possible perfect matchings on a given k-dimensional hypercube. To be completely definite, we match as follows. Given c satisfying Case I, let $\hat{c}$ be the lexicographically greatest representative of the maximum vertex in $\mathrm{HC}([c])$ (computed relative to $\prec$). Since p is prime and $M > 1$, $\hat{c}$ determines a unique representative for every vertex in $\mathrm{HC}([c])$, by splitting any subset of the parts equal to $1 + M$ in $\hat{c}$ that are preceded by an even number of 1s. Now, we obtain a perfect matching of $\mathrm{HC}([c])$ by always splitting or combining at the *first* $M + 1$ occurring in $\hat{c}$. In example (3) above, $\hat{c} = 5311524$, so the matching pairs [5311524] with [14311524] and [53111424] with [143111424].

Case II: All of the y_i are even.
In this case, $[c]$ is part of a k-dimensional hypercube whose vertices differ from $[c]$ by both combinations and splits. We first must address the possibility that $M = 2$. In this event, each 2 must have an even number of 1s preceding it; otherwise we would be in Case I. But since all parts of c are 1 or 2, this implies that p is even, which is a contradiction. So we may assume from now on that $M > 2$ and hence $m = M - 1 > 1$.

The vertex $[c]$ now under consideration in Case II is some non-minimal vertex in one of the hypercubes $\mathrm{HC}([c'])$ previously constructed in Case I. Thus the perfect matching already found for $\mathrm{HC}([c'])$ suffices to match $[c]$, provided we can determine $[c']$ uniquely from $[c]$. Let $B = \{b_1 < b_2 < \cdots < b_q\}$ be the indices for which $c_{b_j} = m$, and let z_j be the number of 1s immediately preceding c_{b_j} (taking wraparound into account). Let $B_{\mathrm{odd}} = \{b_j \in B : z_j \text{ is odd}\}$. We construct a hypercube $\mathrm{HC}([c])$ containing $[c]$ such that $\mathrm{HC}([c])$ has dimension $|A| + |B_{\mathrm{odd}}|$. We obtain the vertices of $\mathrm{HC}([c])$ by independently choosing to split at any of the indices in A or to combine at any of the indices in B_{odd}. One readily checks that $\mathrm{HC}([c]) = \mathrm{HC}([c'])$, where $[c']$ is the minimal element of this hypercube relative to $\prec$. Thus, $[c]$ has already been matched via the matching on $\mathrm{HC}([c'])$ described in Case I. □

Example 25. Let $c = 11142453$. Here $M = 5$ appears one time in c immediately preceded by an even number of 1s, so we are in Case II. Here $m = 4$ also appears once in c preceded by an odd number of 1s, so $\mathrm{HC}([c])$ will be 2-dimensional. To figure out whether we should split the 5 or combine the 14 in order to find the vertex matched to $[c]$, we first find the vertex $[c']$ from Case I with $\mathrm{HC}([c]) = \mathrm{HC}([c'])$. On one hand, splitting at all relevant positions gives $[c'] = [111424143]$. The top vertex in $\mathrm{HC}([c'])$ is found by combining at all relevant positions to get [1152453]. We cyclically rotate to find the lexicographically maximal representative of this vertex, which is $\hat{c} = 5311524$. The first copy of $M = 5$ in this representative determines the parts that get toggled for all vertices in this hypercube. Here, [5311524] matches with [14311524], whereas $[c] = [53111424]$ matches with [143111424]. Thus we match $[c]$ by splitting the 5 rather than combining the parts 14.

Example 26. Let $c = 1115251161515611154315$, so $M = 6$, $m = 5$, $A = \{9, 14\}$, and $B_{\text{odd}} = \{4, 11, 13, 18, 22\}$. We are in Case II, and $[c]$ belongs to a 7-dimensional hypercube obtained by the seven independent choices of either splitting the 6s at positions 9 and 14 or combining the 15s at positions 4, 11, 13, 18 and 22. Splitting at all possible locations in A, we see that $\mathrm{HC}([c]) = \mathrm{HC}([c'])$ where $c' = 111525111515151511154315$. On the other hand, combining at all possible locations in B_{odd} shows that the top vertex of the hypercube is $[11625116666116436]$. The lexicographically maximum representative of this top vertex is $\hat{c} = 66661164361162511$. The initial 6 in $\hat{c}$ corresponds to the 6 at index 9 in c. Therefore, we match $[c]$ by splitting this 6, obtaining the vertex $[11152511151515611154315]$.

The proof of Theorem 24 reveals the following structural property of the graphs $\mathcal{G}_p^*$.

Corollary 27. *For p an odd prime, there exists a collection $\mathcal{H}$ of induced subgraphs of $\mathcal{G}_p^*$ such that (a) each $H \in \mathcal{H}$ is isomorphic to a k-dimensional hypercube for some $k \geq 1$; (b) every vertex in $\mathcal{G}_p^*$ is a vertex of exactly one $H \in \mathcal{H}$.*

7. Results based on Computer Data

This section summarizes some computer investigations of the F-expansions of cyc_G, where G ranges through all subgroups of S_n for small choices of n. For more detailed information, see the SageMath [34] worksheet used to perform most of these computations [35].

Proposition 28. *The standardization approach fully succeeds for every subgroup G of S_3 except $G = \langle(13)\rangle$. For this subgroup, the standardization approach succeeds using $\mathcal{C} = \{123, 132, 213\}$.*

Also notice that $G = \langle(13)\rangle$ is conjugate to a subgroup $H = \langle(12)\rangle$ for which the standardization approach fully succeeds.

Proposition 29. *(a) The standardization approach fully succeeds for the following subgroups of S_4: $\{\mathrm{id}\}$, $\langle(34)\rangle$, $\langle(23)\rangle$, $\langle(12)\rangle$, $\langle(234)\rangle$, $\langle(123)\rangle$, $\langle(34),(12)\rangle$, $\langle(23),(24)\rangle$, $\langle(23),(123)\rangle$, and S_4. (b) The standardization approach succeeds, but not fully, for the following subgroups of S_4: $\langle(24)\rangle$, $\langle(14)\rangle$, $\langle(13)\rangle$, $\langle(134)\rangle$, $\langle(124)\rangle$, $\langle(24),(13)\rangle$, $\langle(23),(14)\rangle$, $\langle(1234)\rangle$, $\langle(34),(134)\rangle$, $\langle(24),(124)\rangle$, and $\langle(12)(34),(24)\rangle$.*

Every conjugacy class of subgroups of S_4 contains a representative subgroup where the standardization approach succeeds, with the sole exception of the conjugacy class of $G = \langle(12)(34)\rangle$. For this G, (17) holds algebraically with $\mathcal{D} = \{\emptyset, \{2\}, \{1,3\}, \{1,2,3\}\}$.

There are 19 conjugacy classes of subgroups of S_5. Ten of these classes have representatives where the standardization approach fully succeeds, and four of the other classes have representatives where the standardization approach succeeds. There are five other classes where (17) holds algebraically for some choice of $\mathcal{D}$.

Finally, we have some isolated experimental results indicating that the *set* $\mathcal{C}$ in (1) can sometimes be taken to be a *subgroup* of S_n. Here are some examples when $n = 7$: for $G = A_7$ we may take $\mathcal{C} = \langle(17)(26)(35)\rangle$; for $G = \langle(23)(45)(67)\rangle$ we may take $\mathcal{C} = A_7$; for $G = S_6' \times S_1'$ we may take $\mathcal{C} = \langle(1234567)\rangle$; and for $G = A_6' \times S_1'$ we may take $\mathcal{C} = \langle(1234567), (27)(36)(45)\rangle$. In general, these subgroups $\mathcal{C}$ do not always yield a bijective proof via the standardization approach, since (10) need not hold.

Our experiments show that the standardization approach succeeds much more often than one might expect. We leave it as an open problem to characterize those subgroups of S_n (or conjugacy classes of subgroups) for which the standardization approach succeeds or fully succeeds.

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