A combinatorial variant of Sylvester’s four-point problem?

Greg Warrington

The University of Vermont





MathFest: Gems of Combinatorics Session

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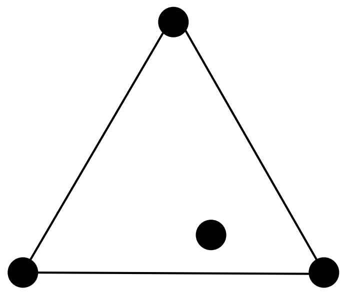
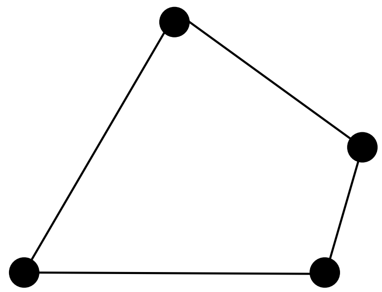
Sylvester's problem



vs.



Direct calculation

Average area of a triangle picked from a disk D

$$\bar{A} = \frac{1}{A(D)^3} \iint_{(x_1,y_1)} \iint_{(x_2,y_2)} \iint_{(x_1,y_1)} \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} dx_i \, dy_i.$$

Probability of forming a reentrant quadrilateral

$$4\frac{\bar{A}}{A(D)}$$

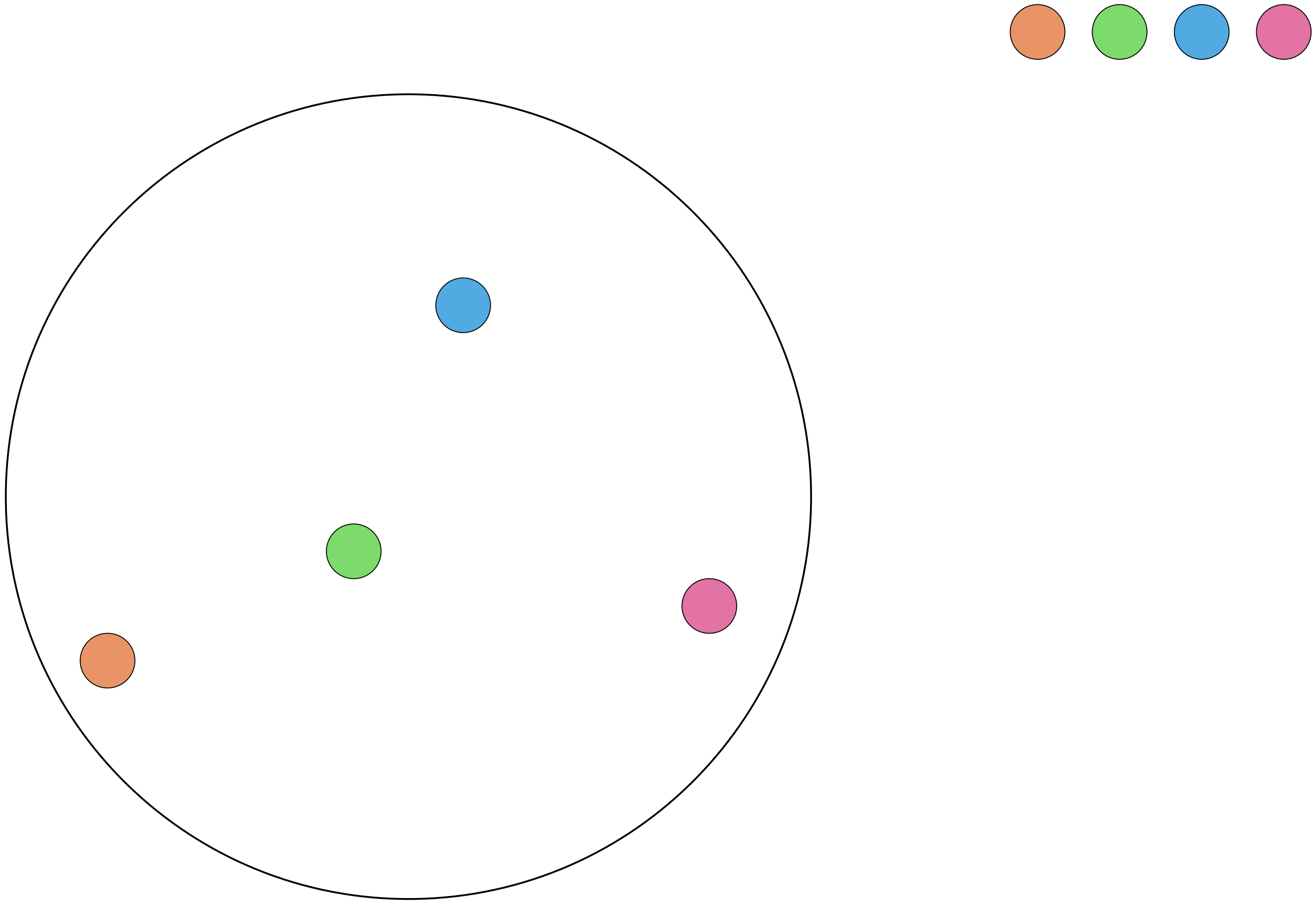
Exact answers

Region	Probability	Approx.
Disk	$\frac{35}{12\pi^2}$	0.29552
Square	$\frac{11}{36}$	0.30556
Triangle	$\frac{1}{3}$	0.33333

W. Woolhouse 1867, W. Blaschke 1917

(Nonconvex regions) D. Singer 1971, unpublished

Allowable sequences



R. Perrin 1882, Goodman-Pollack 1980

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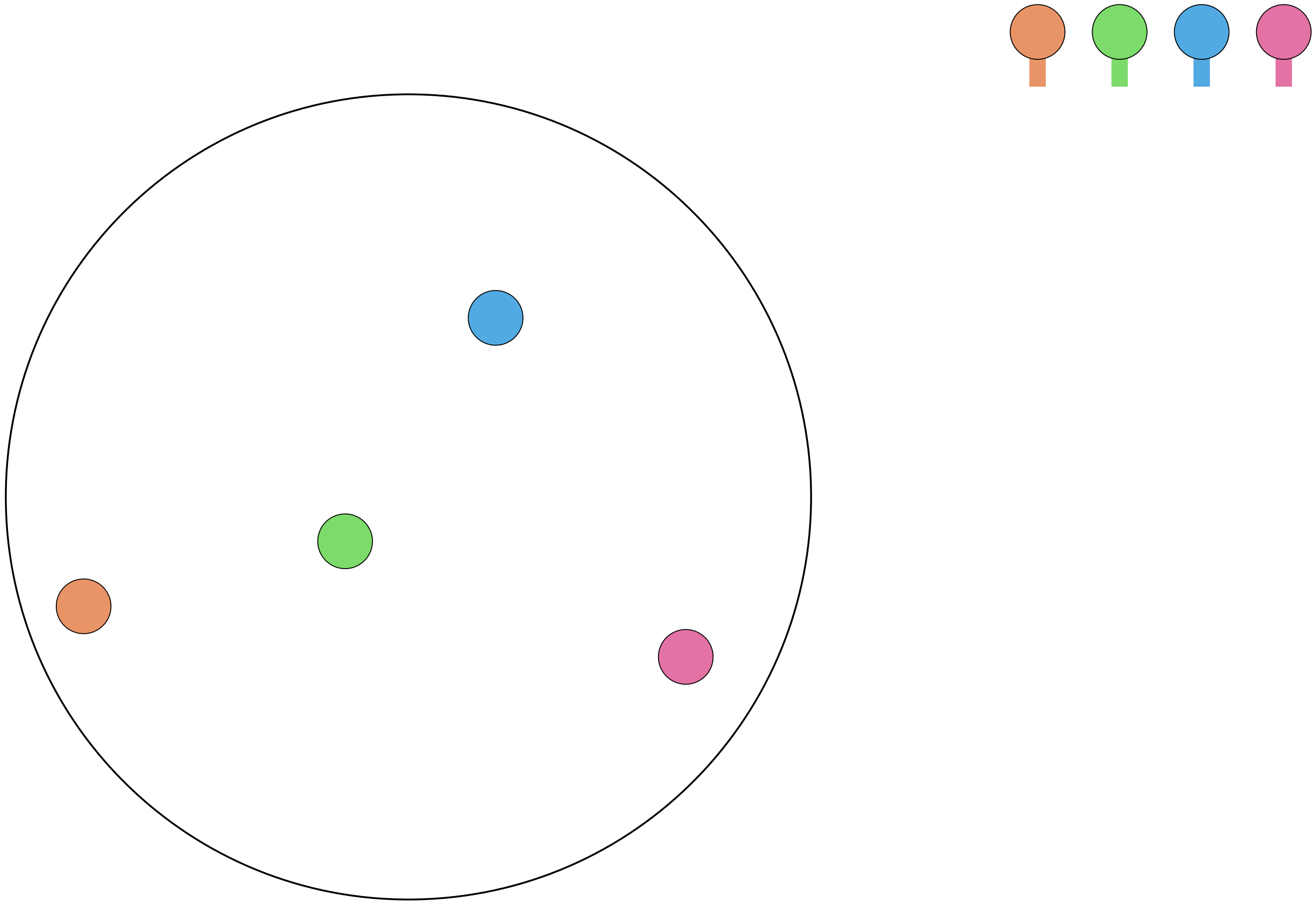

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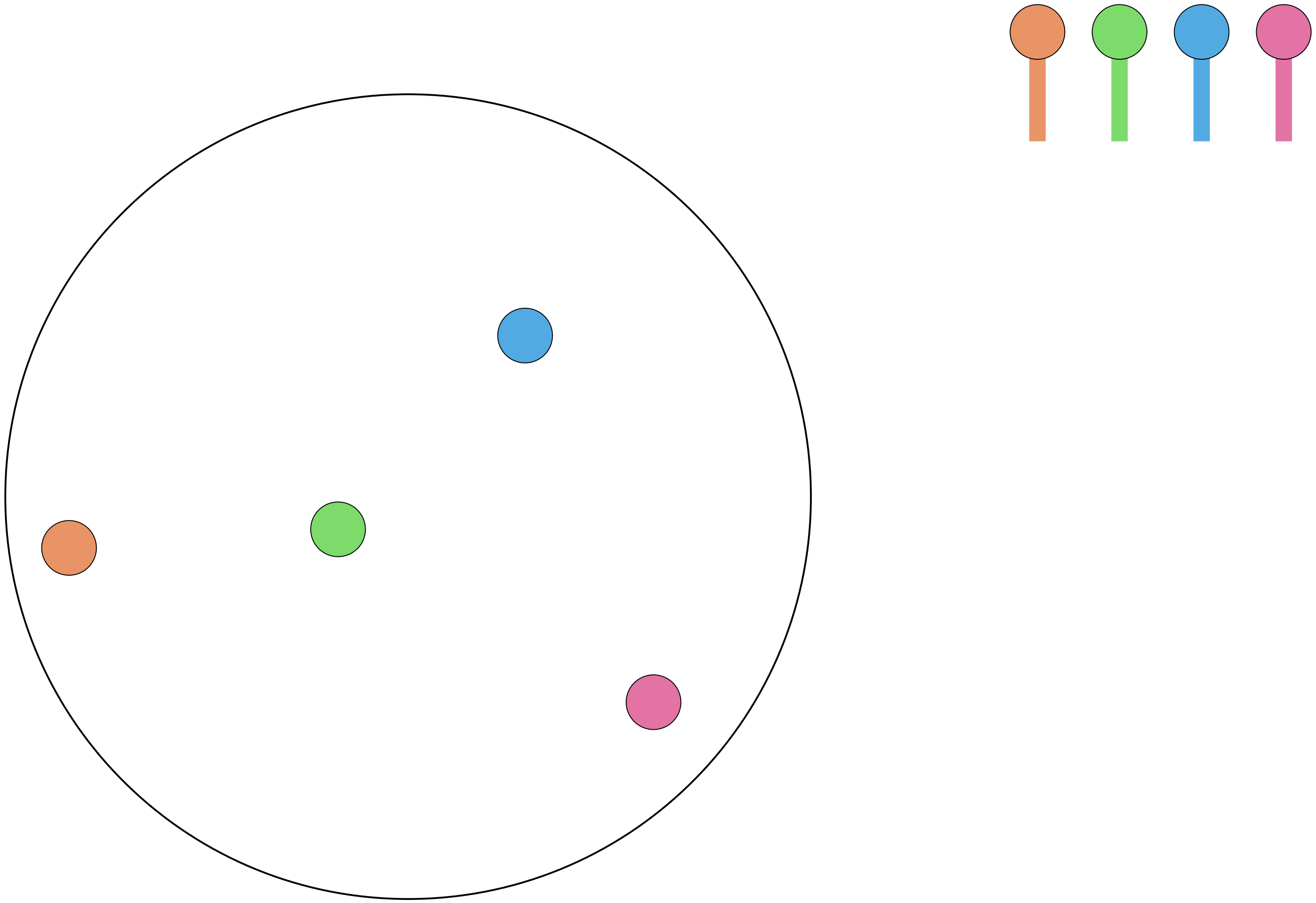

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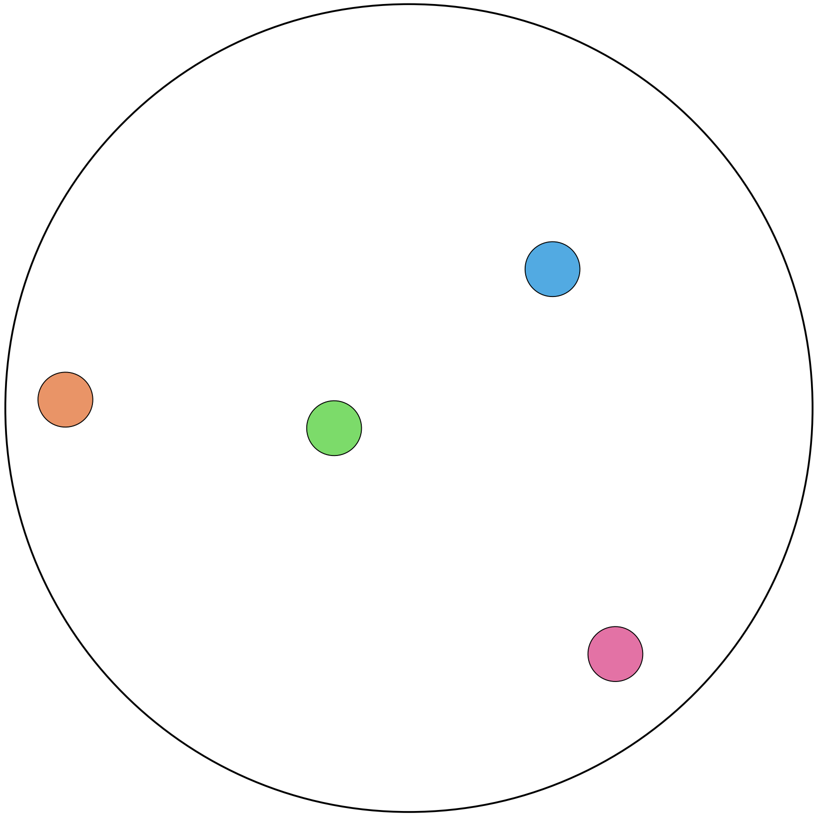

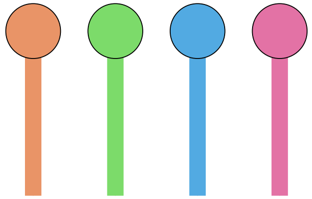

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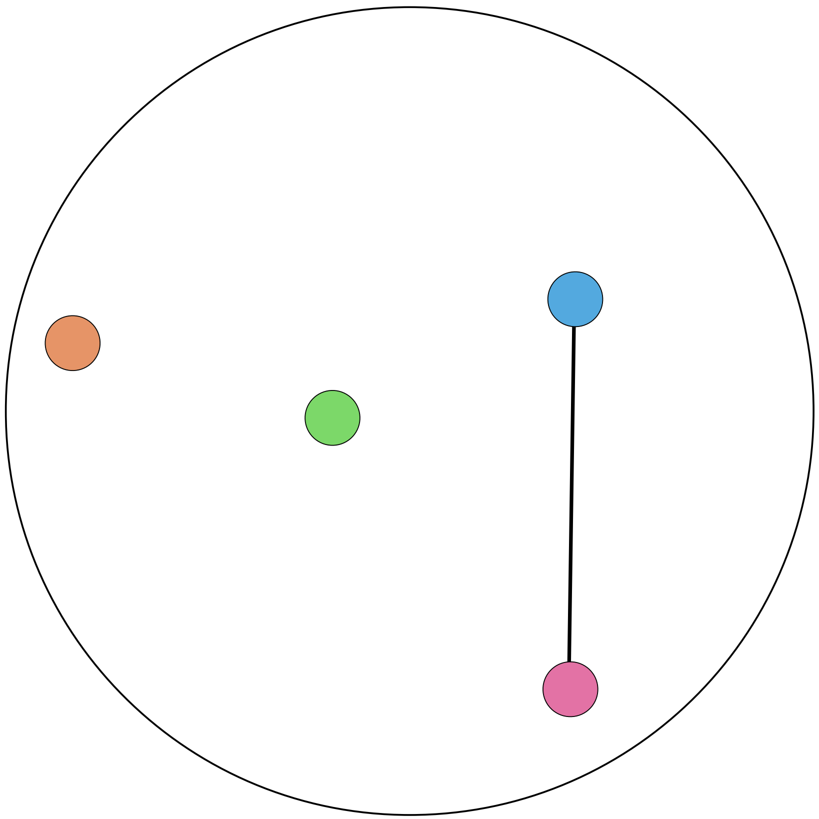

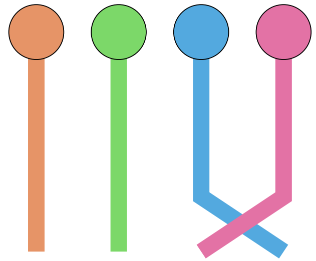

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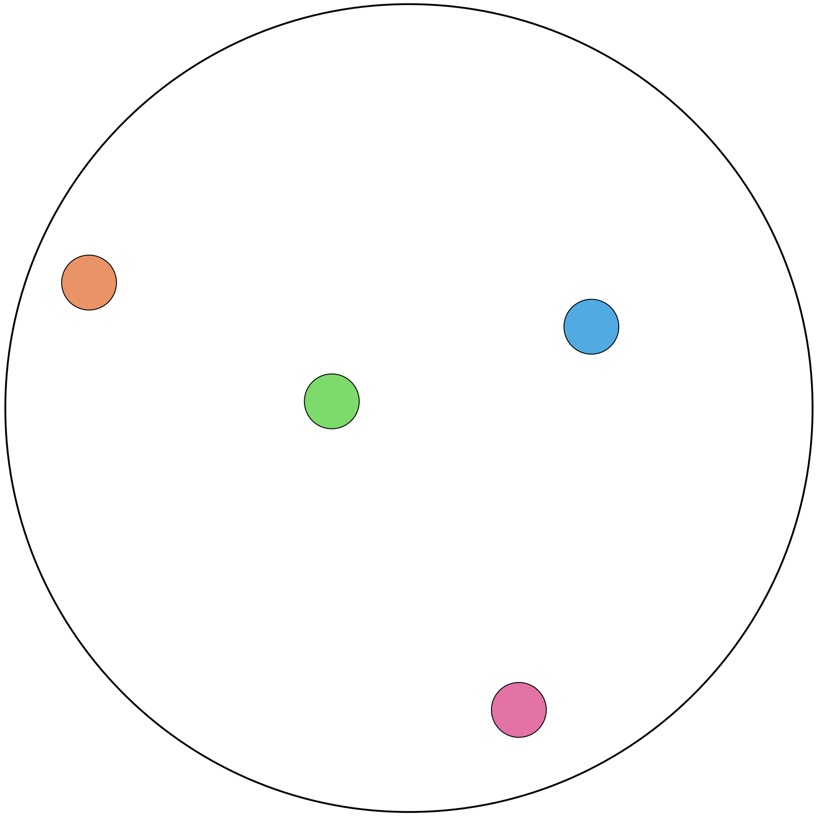

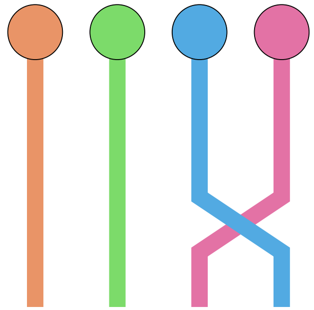

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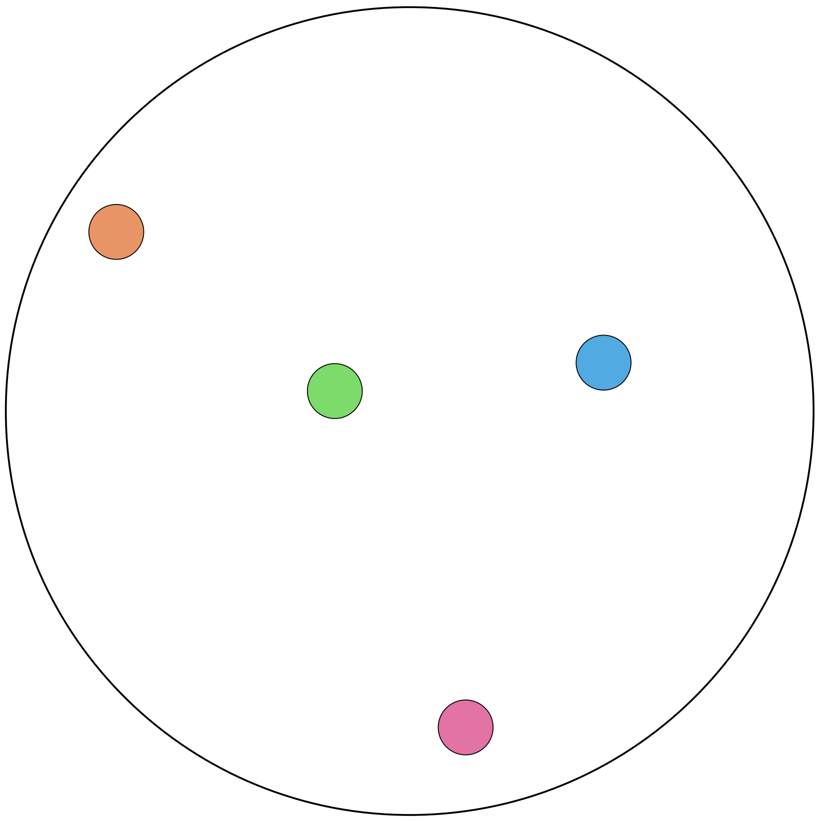

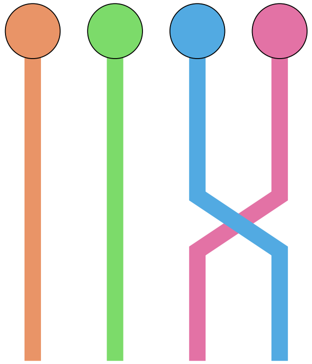

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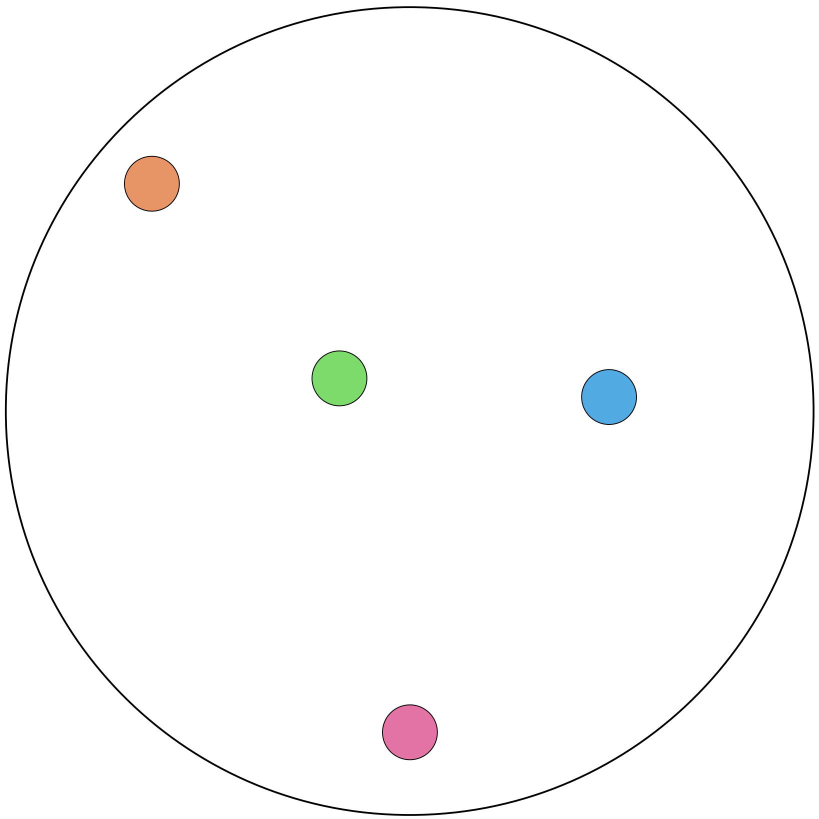

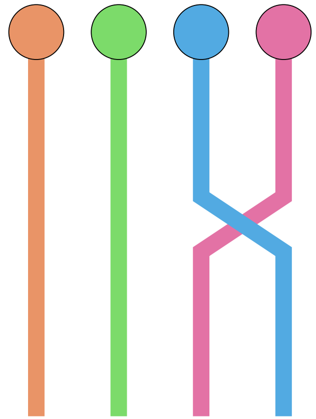

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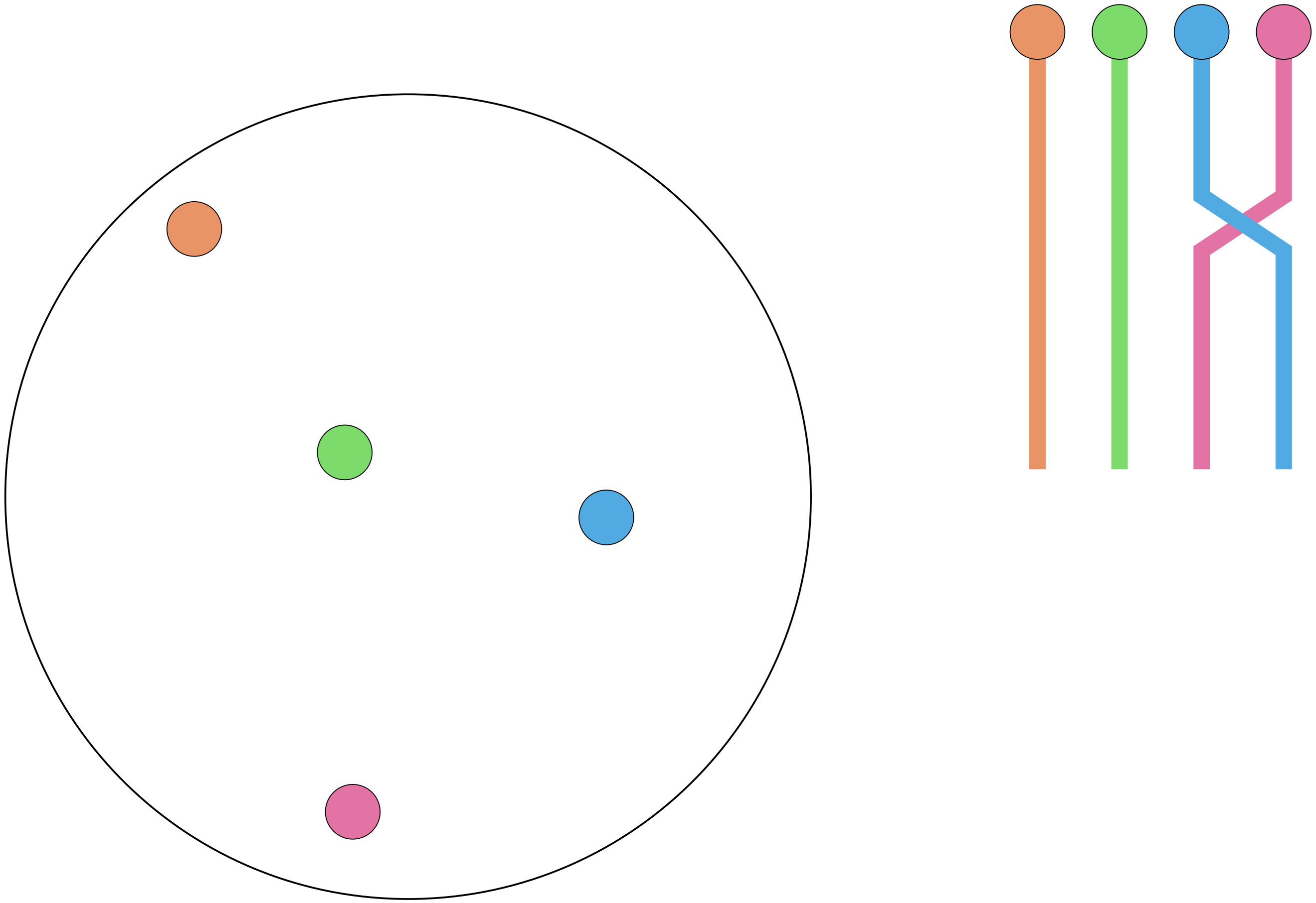

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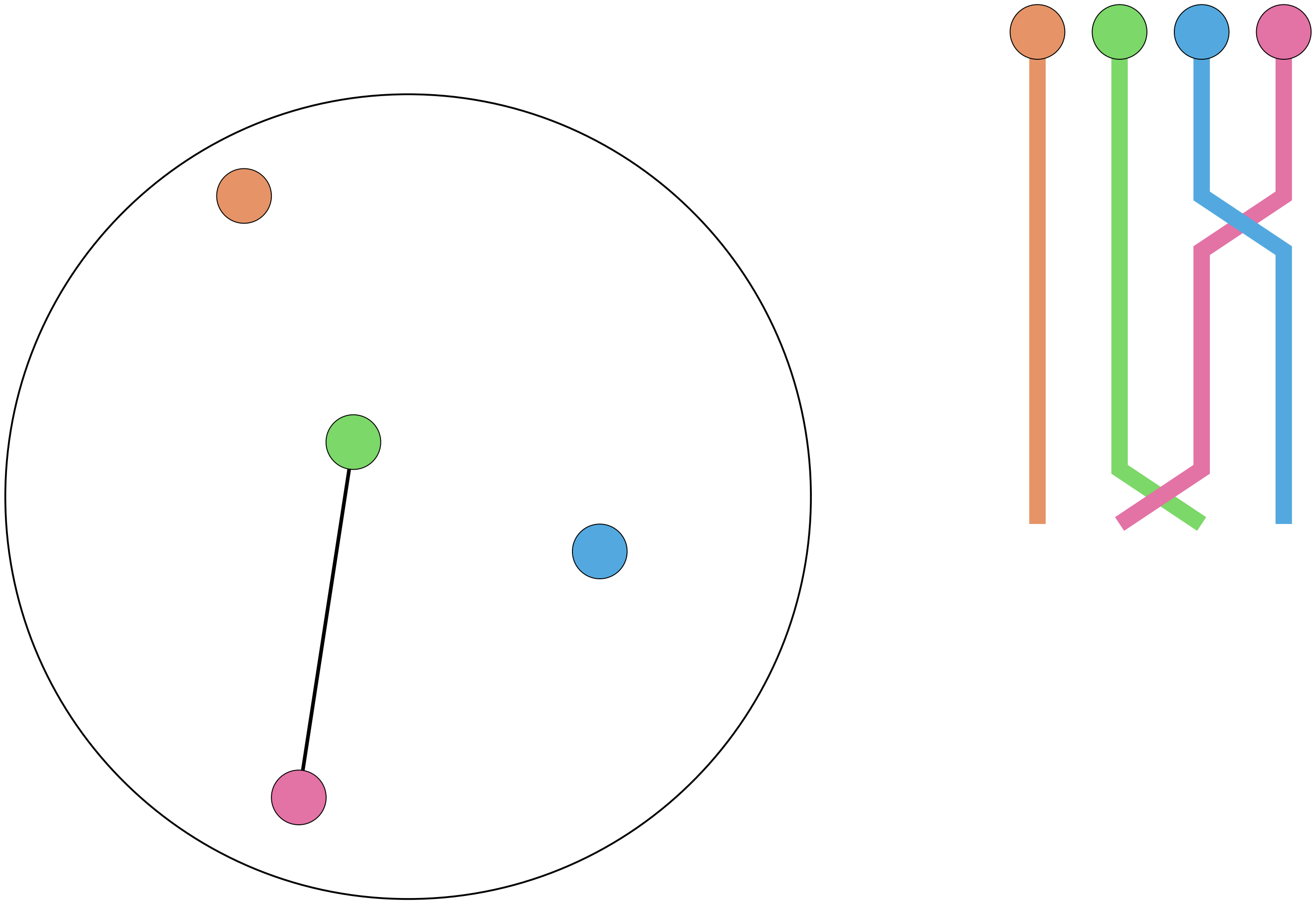

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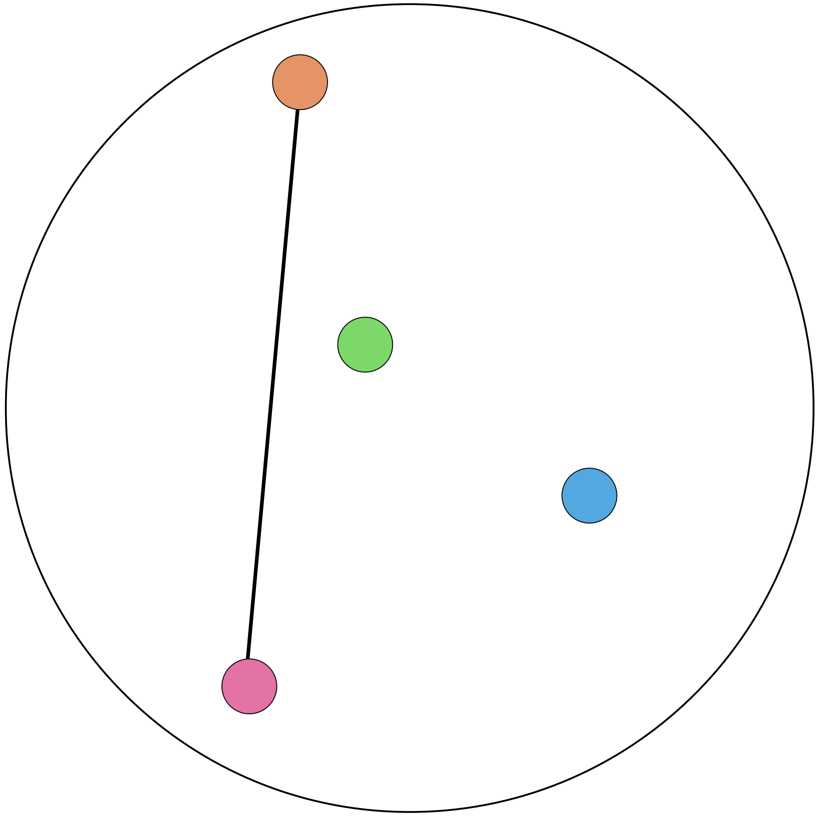

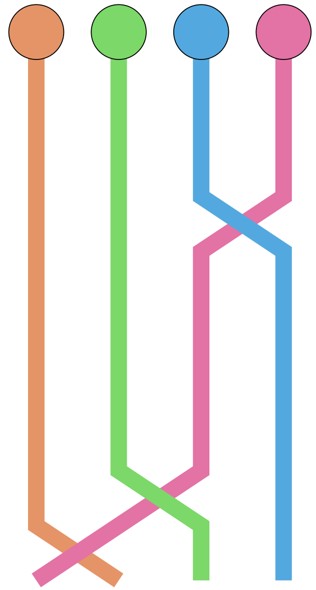

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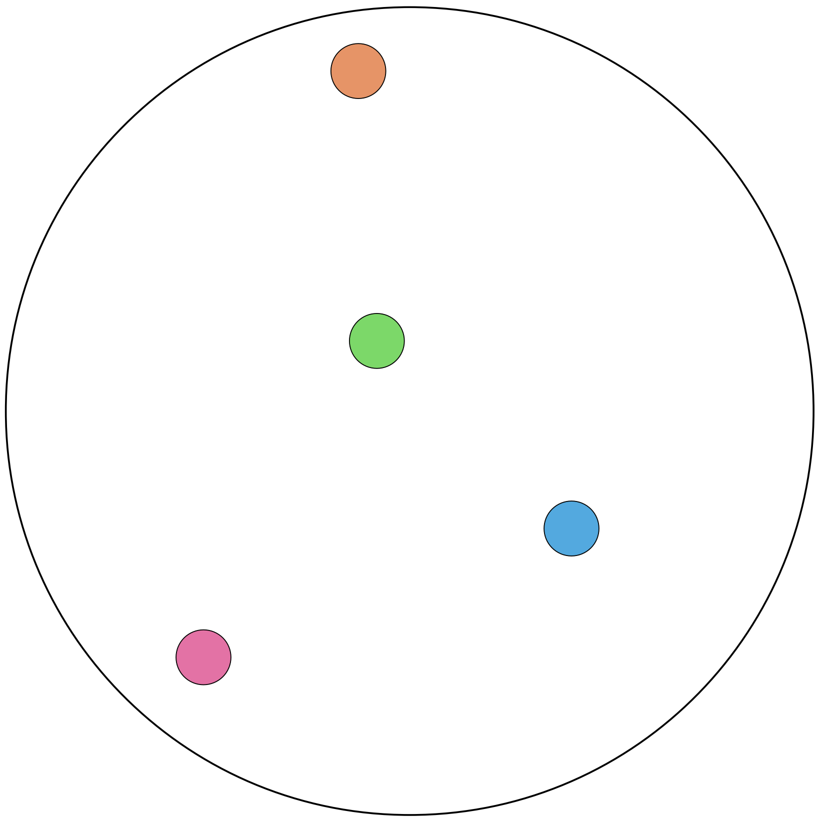

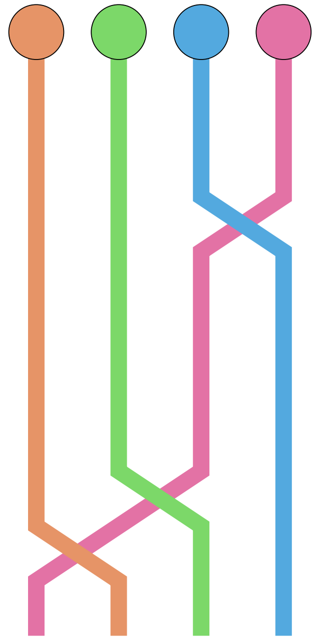

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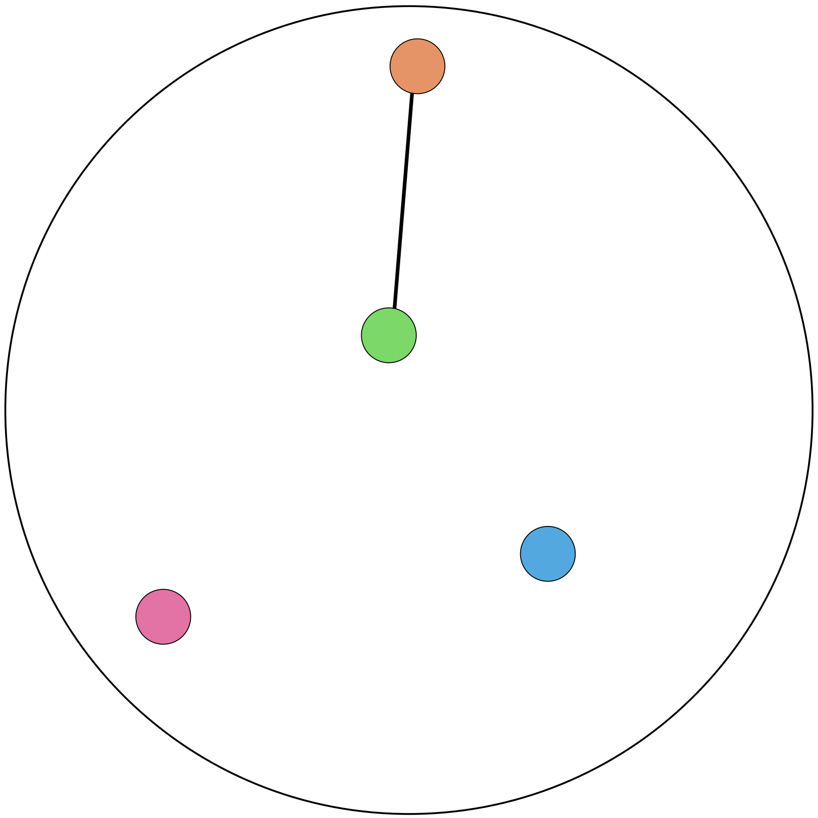

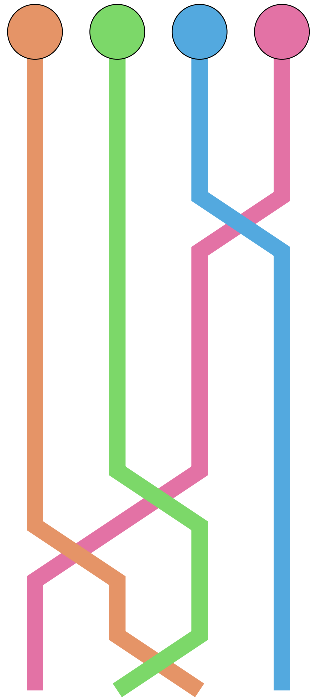

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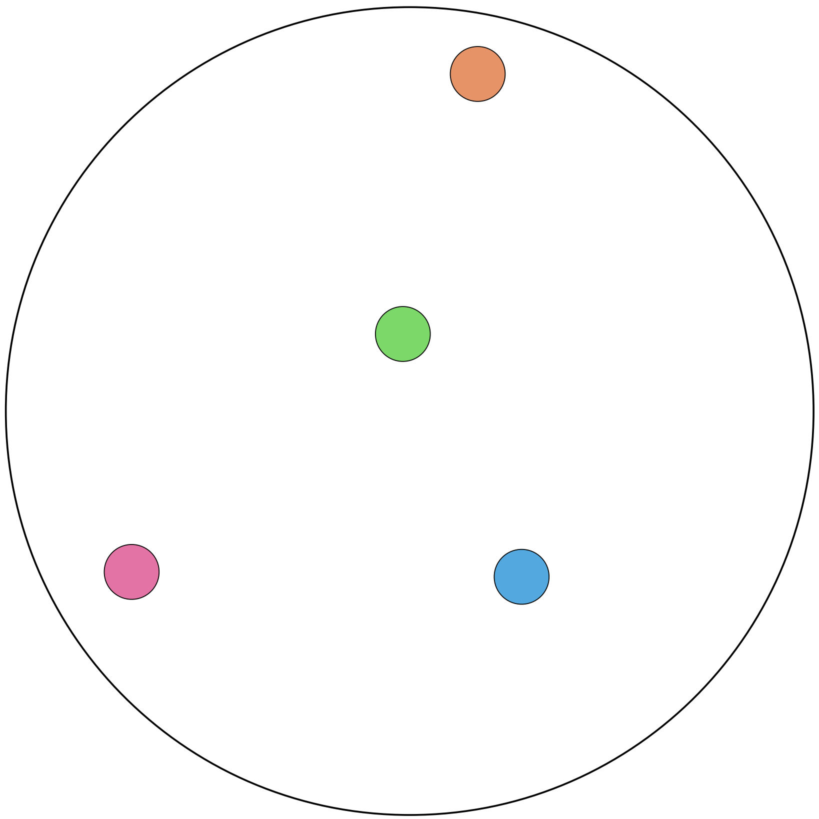

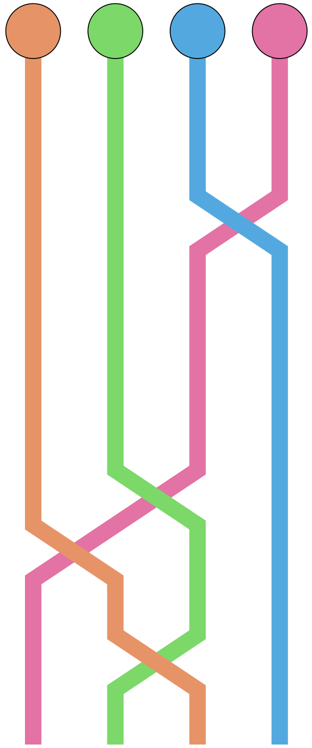

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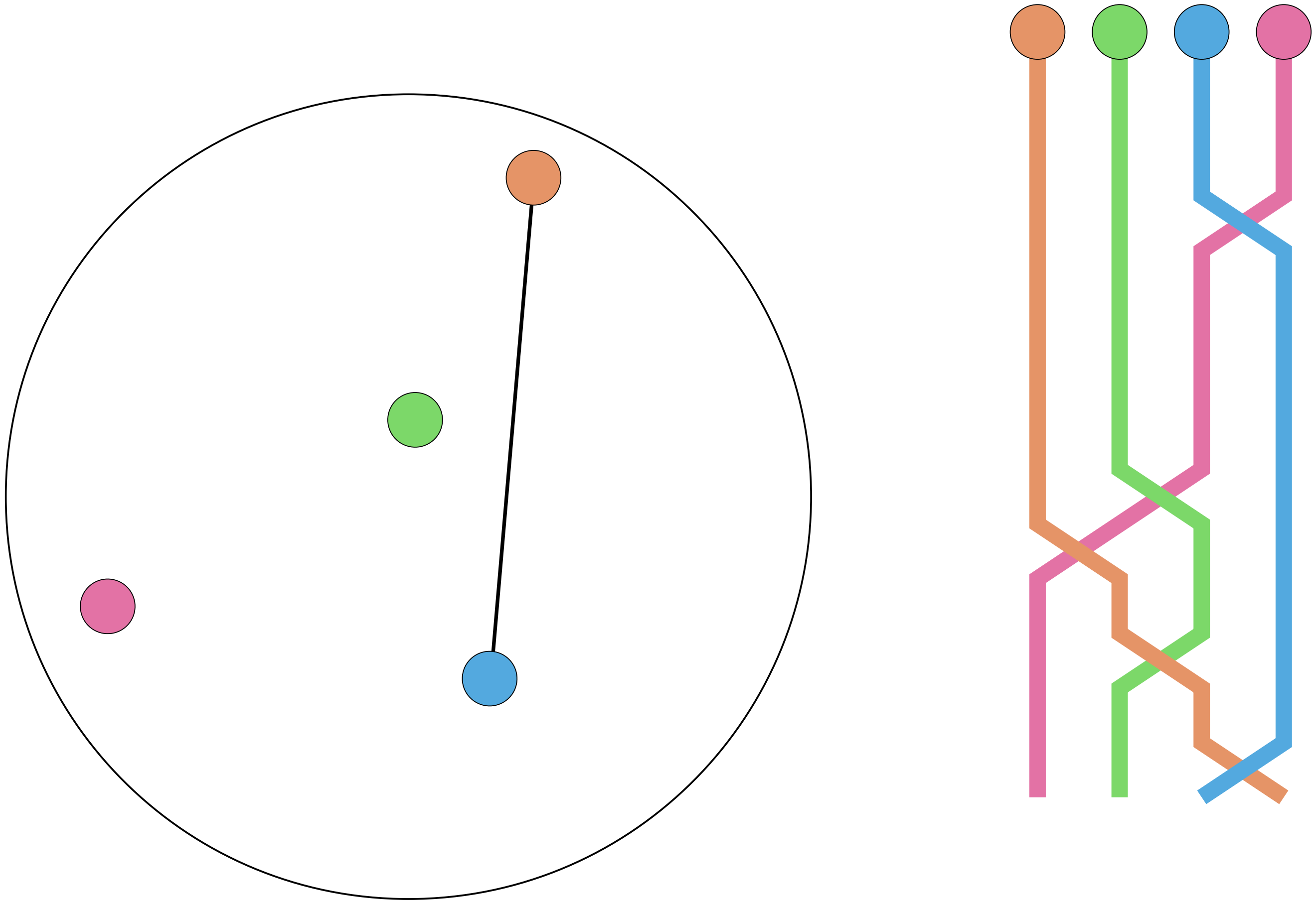

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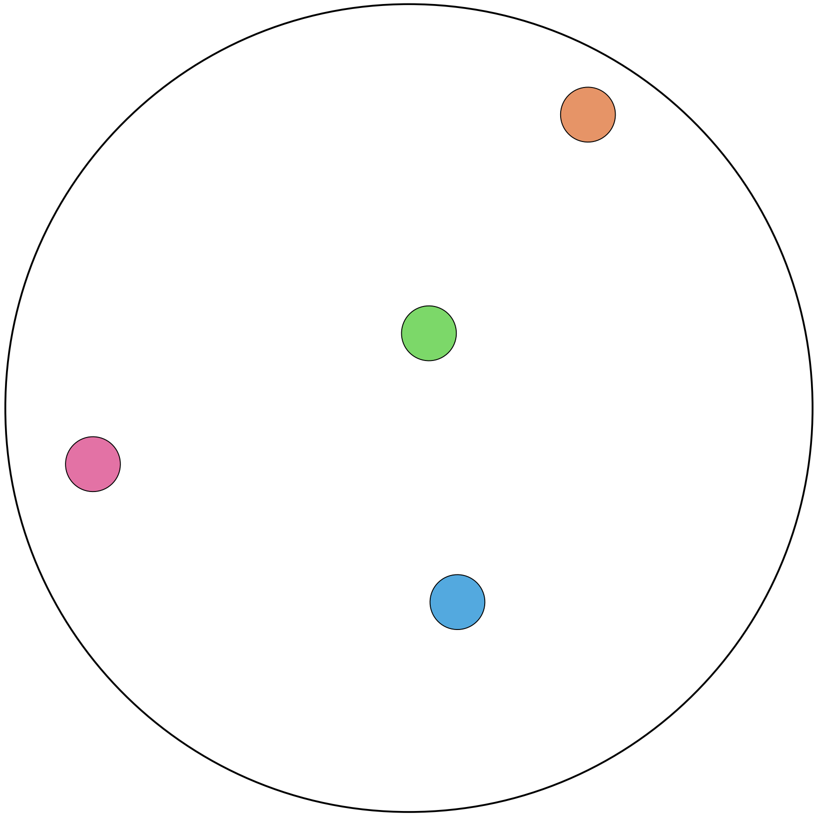

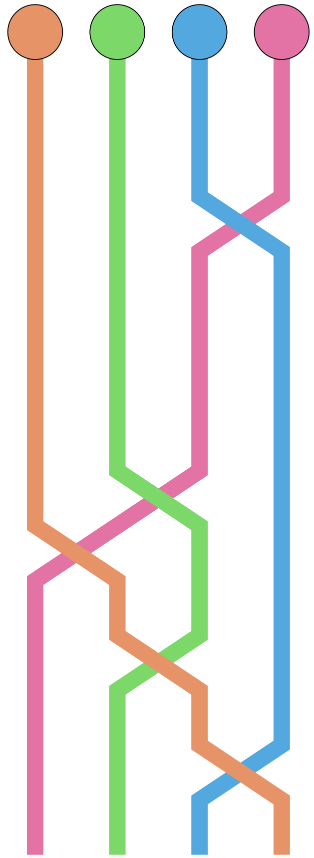

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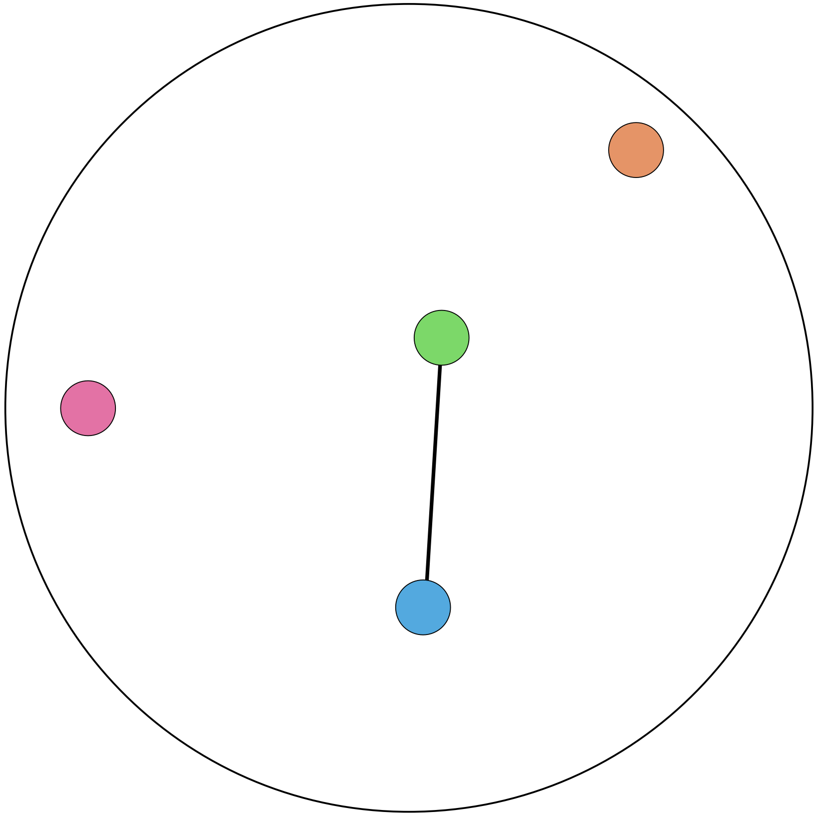

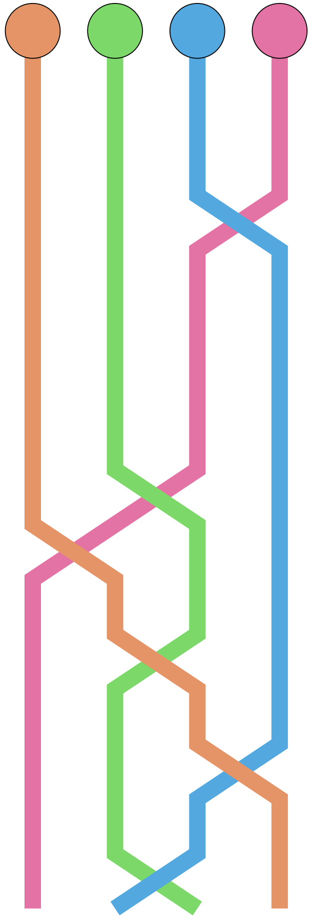

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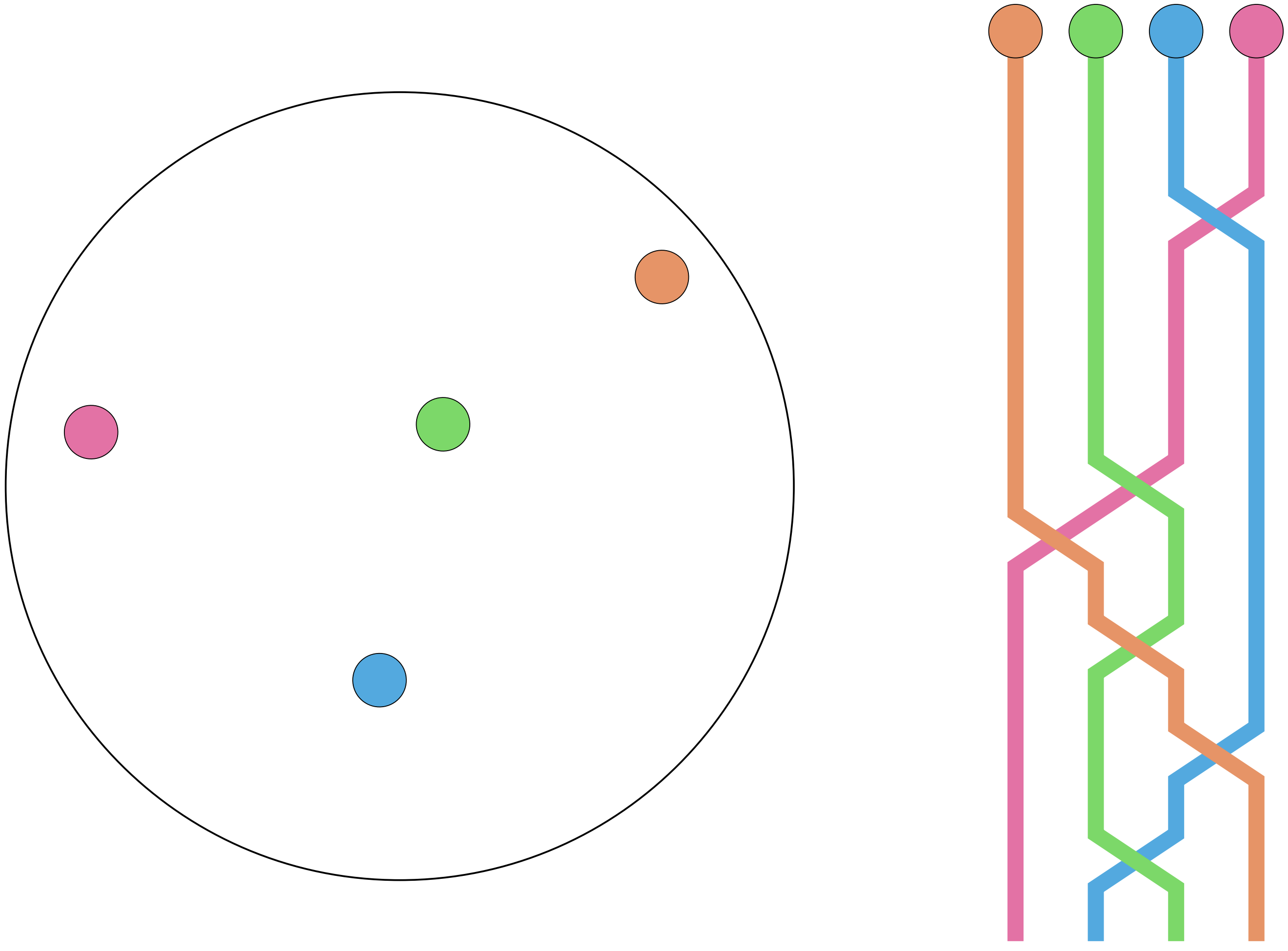

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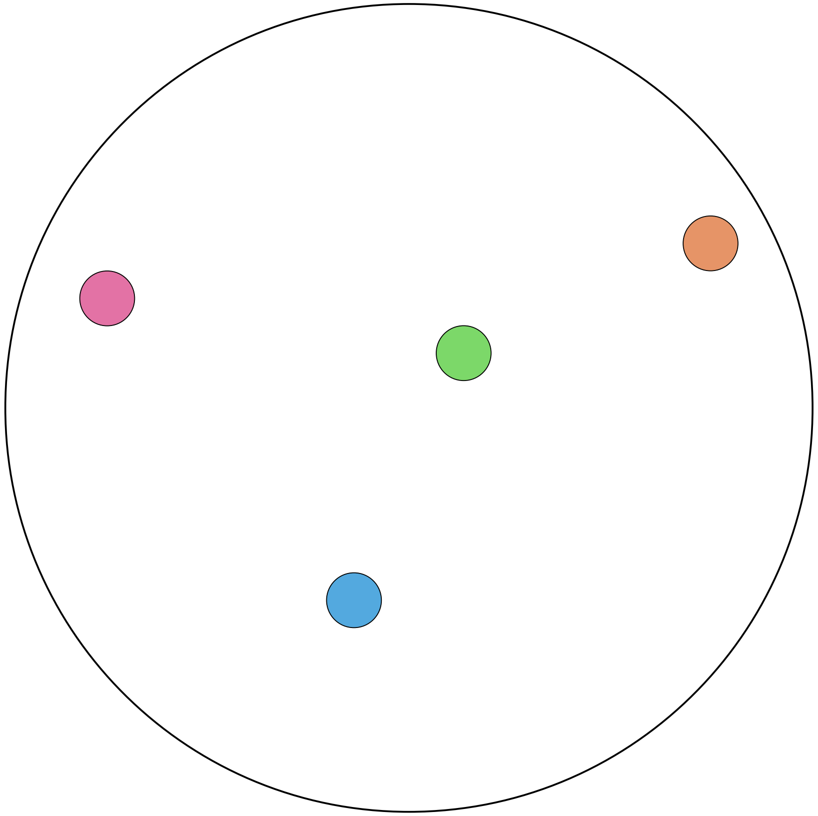

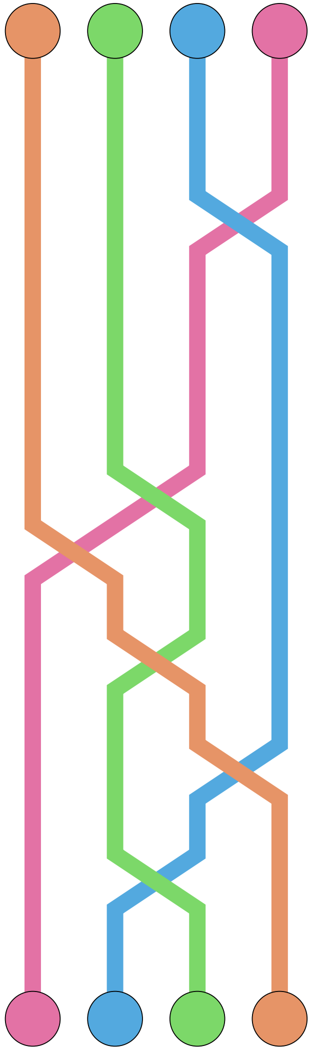

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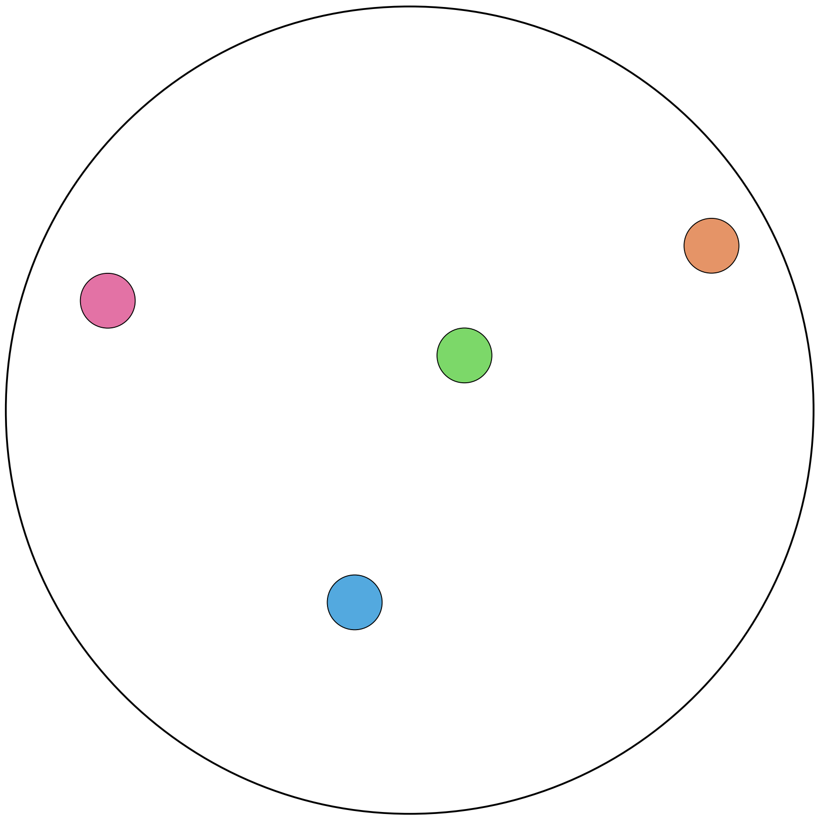

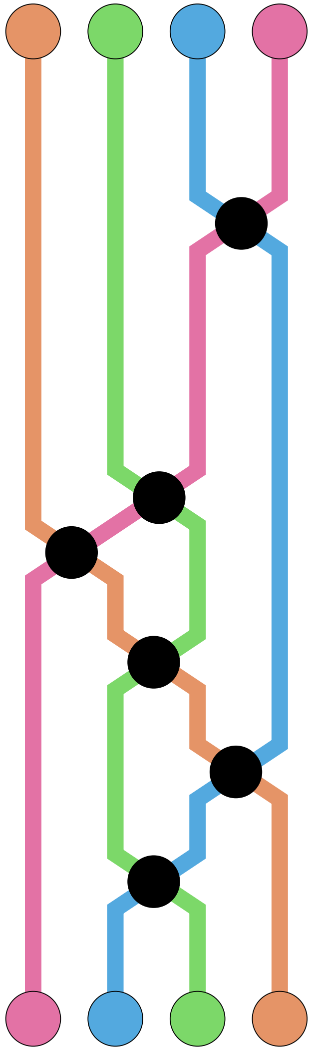

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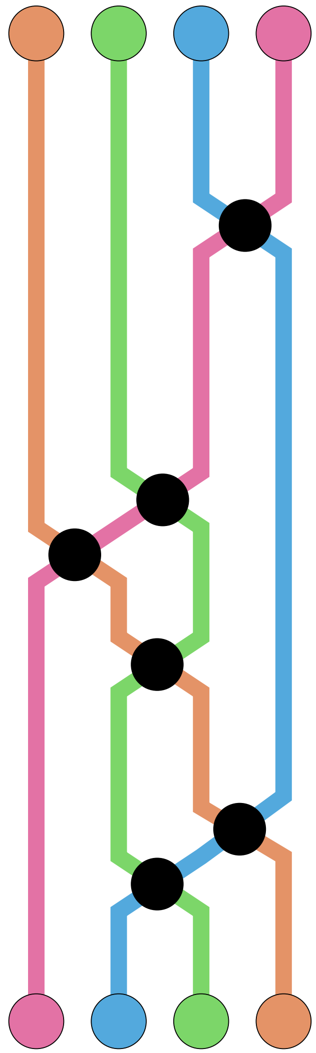

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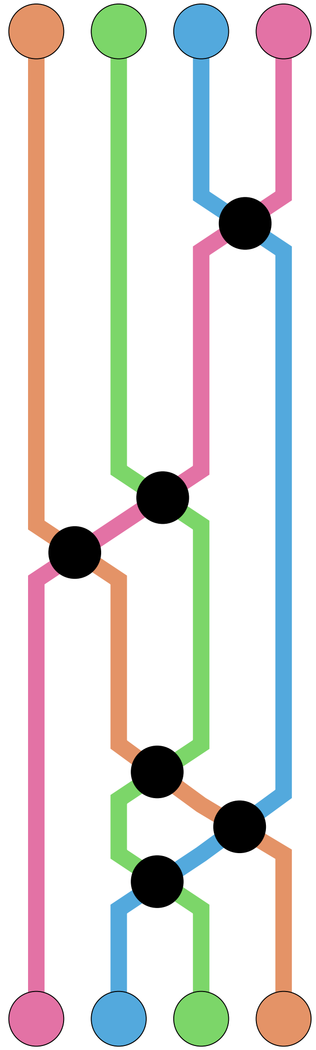

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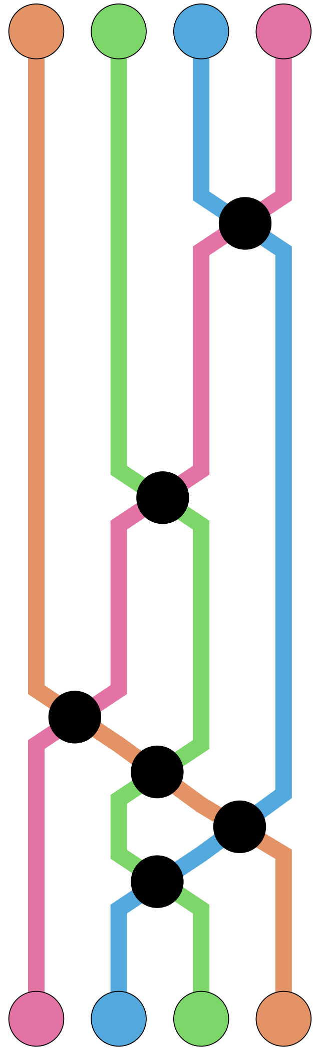

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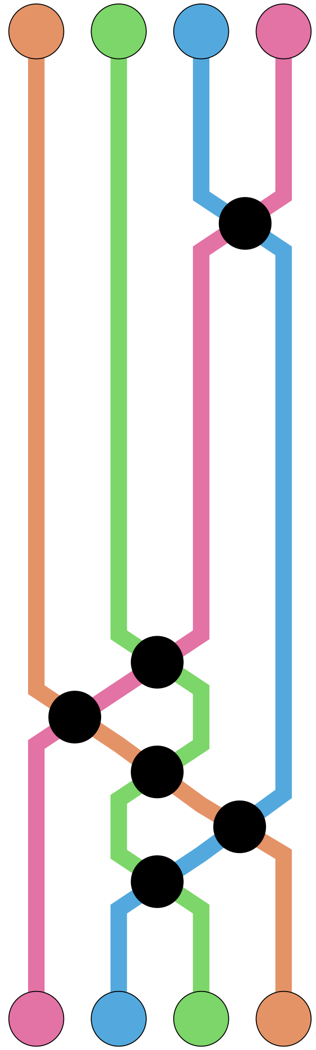

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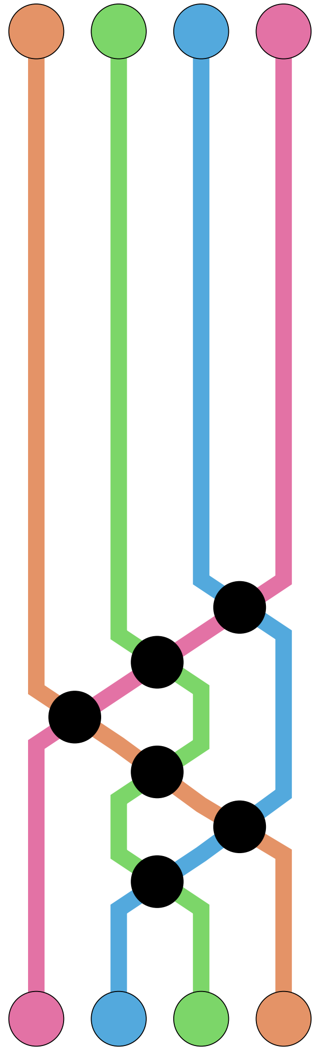
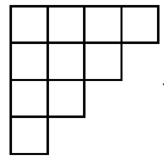


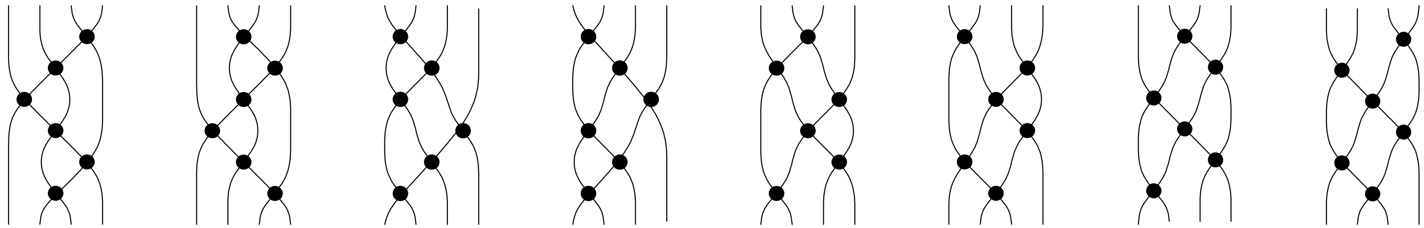
Tableaux

Theorem[Stanley '84] The number of allowable sequences on n points is the number of Standard Young Tableaux of shape $(n-1, n-2, \ldots, 2, 1)$ (e.g., ):

$$\frac{\binom{n}{2}!}{1^{n-1}3^{n-2}\cdots(2n-3)^1}.$$

Frequencies (1000 runs)

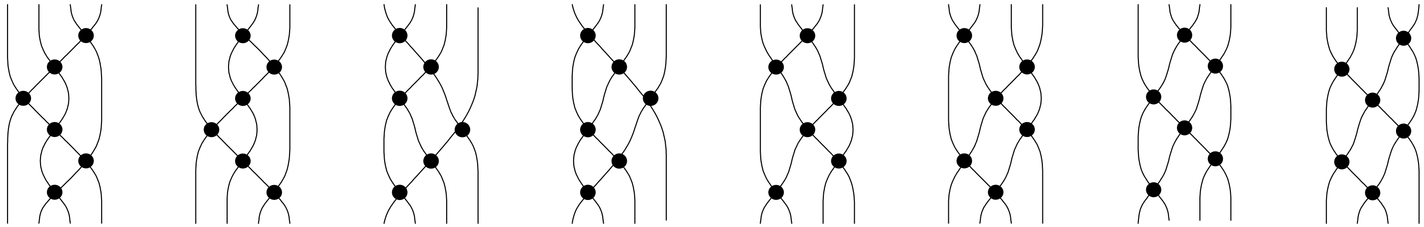
Reentrant



296	○	148	148	134	134	129	129	089	089
306	□	153	153	139	139	125	125	083	083
333	◿	211	122	119	148	090	153	060	097
002	●	001	001	248	248	125	125	125	125

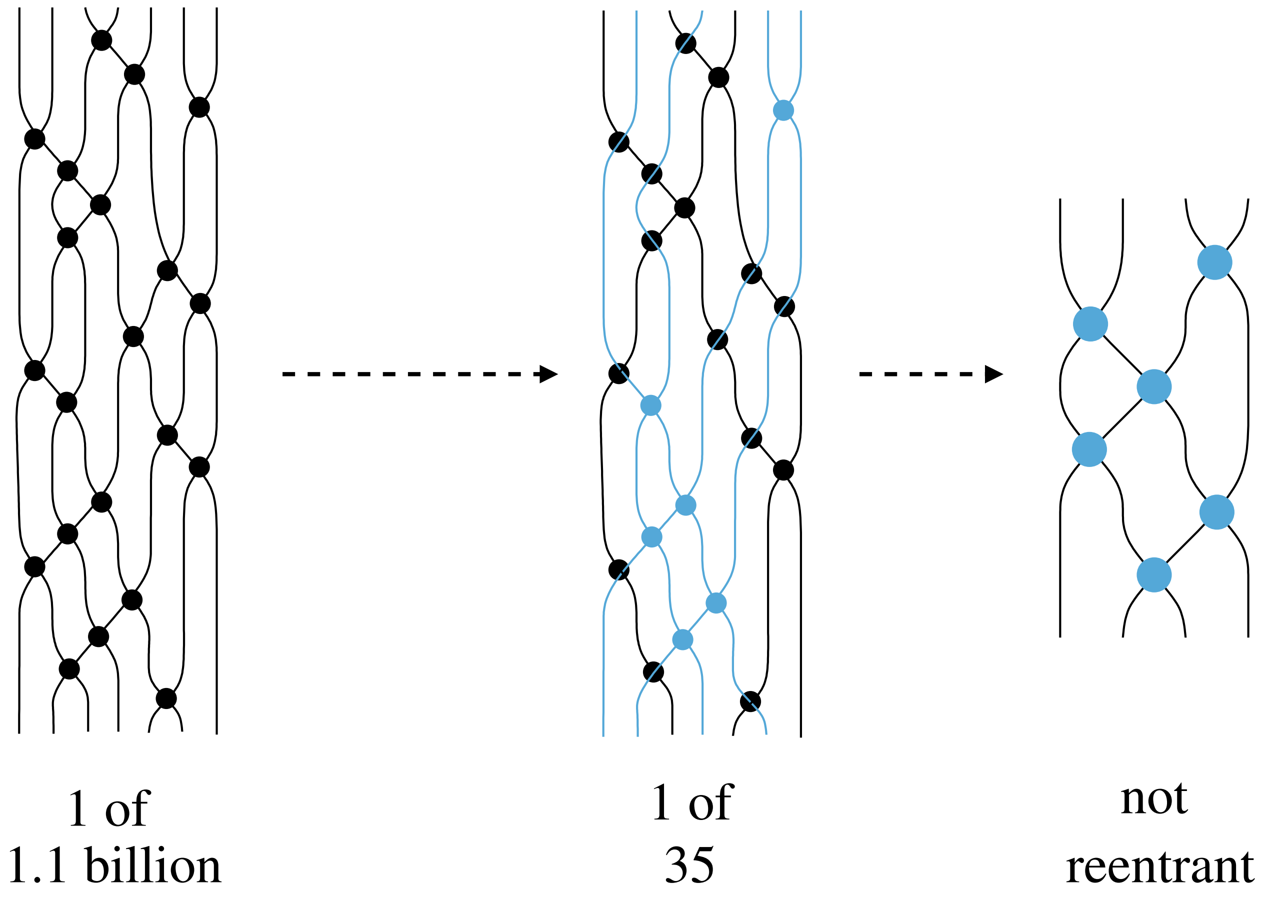
Frequencies (1000 runs)

Reentrant

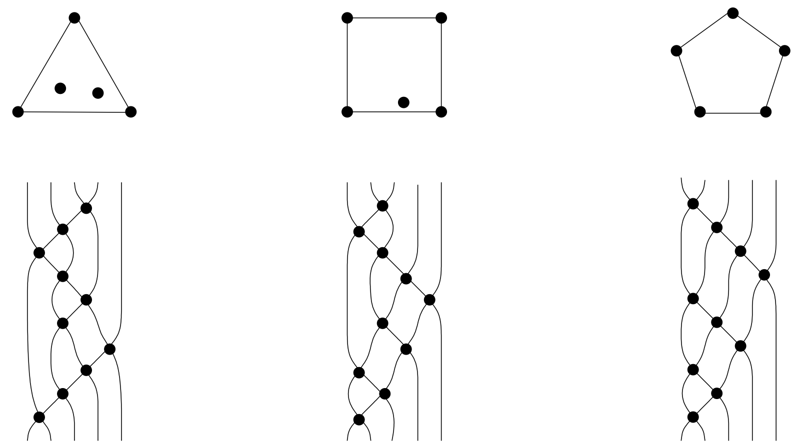


296	○	148	148	134	134	129	129	089	089
306	□	153	153	139	139	125	125	083	083
333	◺	211	122	119	148	090	153	060	097
002	●	001	001	248	248	125	125	125	125
250		125	125	125	125	125	125	125	125

Combinatorial version

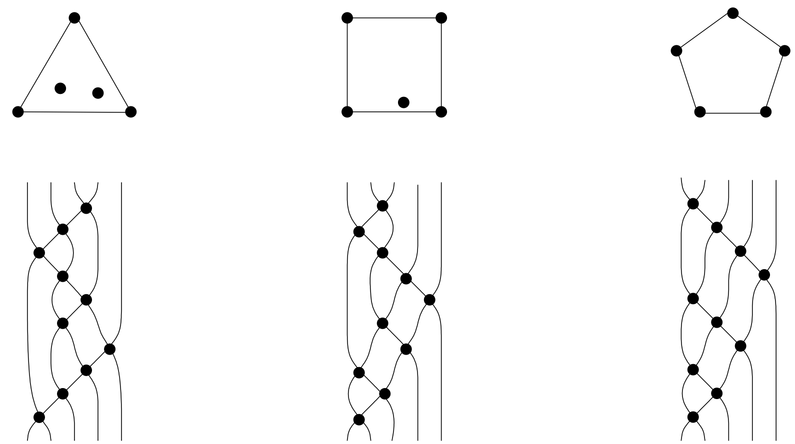


Five points



# reentrant	4	2	0
Size of class	40	400	328

Five points



# reentrant	4	2	0
Size of class	40	400	328

Probability of being reentrant

$$\frac{40 \cdot 4 + 400 \cdot 2 + 328 \cdot 0}{\binom{5}{4}\,(40 + 400 + 328)} = \frac{1}{4}$$

Six, seven and eight points

6 points $$\frac{1{,}098{,}240}{\binom{6}{4}292{,}864} = \frac{1}{4}$$

7 points $$\frac{9{,}631{,}498{,}240}{\binom{7}{4}1{,}100{,}742{,}656} = \frac{1}{4}$$

8 points $$\frac{850{,}653{,}924{,}556{,}800}{\binom{8}{4}48{,}608{,}795{,}688{,}960} = \frac{1}{4}$$

Conjecture (true for $n = 4, 5, 6, 7, 8$)

The probability that 4 points chosen from $n \geq 4$ points are "reentrant" is $\frac{1}{4}$.

Next step

Show that number of reentrant four tuples arising from all allowable sequences on 9 points is

$$\mathbf{921{,}638{,}560{,}382{,}142{,}382{,}080}$$

9 points $\quad \dfrac{?}{\binom{9}{4}\mathbf{29{,}258{,}366{,}996{,}258{,}488{,}320}} = ?$

Culs-de-sac


Equivalence Classes
SYT
Oriented Matroids
Induction Junction
RSK
3:1 map
Expec–tation