An Overview of Kazhdan-Lusztig Polynomials

Greg Warrington
University of Pennsylvania
www.math.upenn.edu/~gwar
gwar@math.upenn.edu

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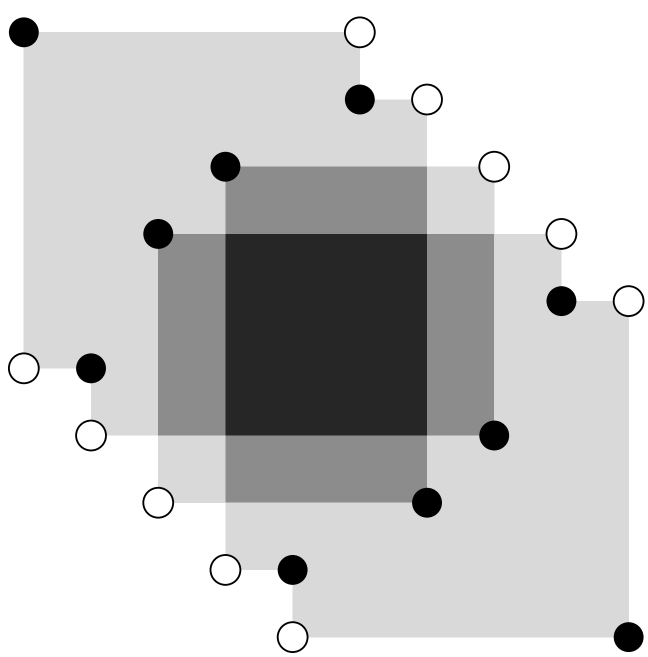


K-L polynomials

For any Coxeter group, W, Kazhdan and Lusztig defined polynomials $P_{x,w}$ for each pair $x, w \in W$.

1. Combinatorics
   - Intrinsic proofs of properties
2. Geometry
   - Singularities of Schubert varieties
   - Poincaré polynomials
3. Representation theory
   - Representations of Hecke algebras
   - Multiplicities in Verma modules

Coxeter groups

A **Coxeter group**, $\boldsymbol{W}$, is a group with presentation

$$W = \langle s_\alpha \in \Delta \mid (s_\alpha s_\beta)^{m_{\alpha,\beta}} = 1 \rangle$$

where

- Δ is some finite indexing set,
- $m_{\alpha,\alpha} = 1$ for all $\alpha \in \Delta$, and
- $m_{\alpha,\beta} = m_{\beta,\alpha} \in \{1, 2, \ldots\} \cup \infty$ for all $\alpha, \beta \in \Delta$.

Example:

S_n: The symmetric group on n letters.

- $\Delta = \{1, 2, \ldots, n-1\}$,
- $s_i \leftrightarrow (i, i+1)$,
- $m_{i,j} = 3$ if $|i-j| = 1$, and
- $m_{i,j} = 2$ if $|i-j| > 1$.

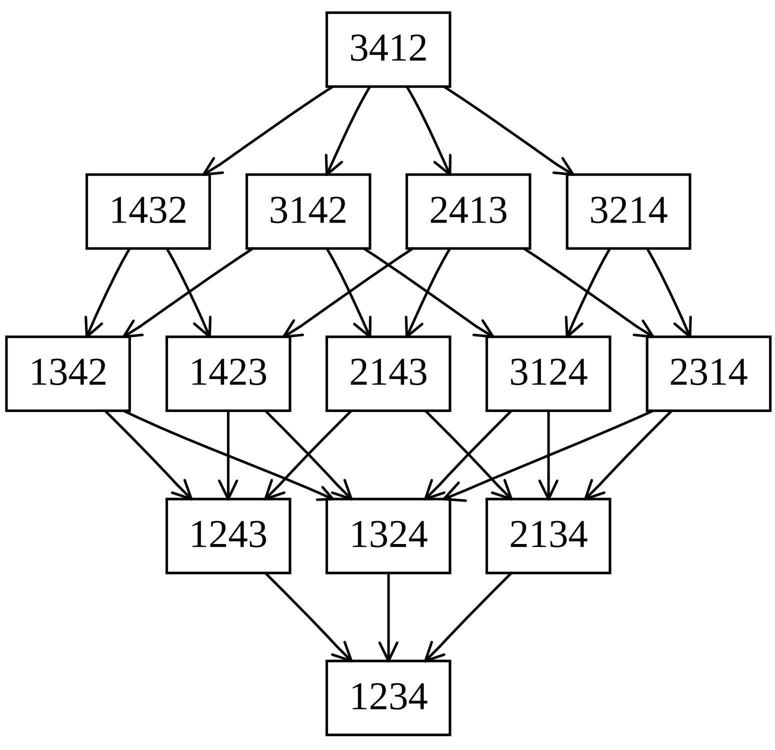
Ranking the poset

The **length** of $x \in W$, denoted $l(x)$, is the minimum k such that we can write $s_{i_1} \cdots s_{i_k}$.

Example:
For S_n, $l(w) = \#\{i < j \mid w(i) > w(j)\}$.

The **Bruhat-Chevalley order** is the transitive closure of $v < vt$ when

1. $t = w s_\alpha w^{-1}$ for some $w \in W$, and
2. $l(vt) = l(v) + 1$.



Hecke algebras

The Hecke algebra, $\mathcal{H}(W)$, associated to the Coxeter group W is the $\mathbb{Z}[q^{1/2}, q^{-1/2}]$-algebra with basis $\{T_w\}_{w\in W}$ and multiplication:

$$T_sT_w = \begin{cases} T_{sw}, & \text{if } l(sw) = l(w)+1, \\ qT_{sw} + (q-1)T_w, & \text{if } l(sw) = l(w)-1. \end{cases}$$

Define an involution $\overline{\cdot}$ on $\mathcal{H}(W)$ as the linear extension of $q \mapsto q^{-1}$ and $T_w \mapsto (T_{w^{-1}})^{-1}$.

Theorem 1 (K-L). *There is a unique basis $\{C'_w\}_{w\in W}$ of $\mathcal{H}(W)$ such that*

1. $\overline{C'_w} = C'_w$, *and*
2. $C'_w = q^{\frac{-l(w)}{2}} \sum_{x\leq w} P_{x,w}T_x$,

where $P_{x,w} \in \mathbb{Z}[q]$, $P_{w,w} = 1$ and

$$\deg(P_{x,w}) \leq \frac{l(w)-l(x)-1}{2}$$

when $x < w$.

Properties of Kazhdan-Lusztig (KL) polynomials

1. $P_{x,w} = 0$ if $x \not\leq w$.

2. $P_{x,w}(0) = 1$ if $x \leq w$.

3. $P_{sx,w} = P_{x,w}$ if $sw < w$.

Let $\mu(x, w)$ be the coefficient of $q^{\frac{l(w)-l(x)-1}{2}}$ in $P_{x,w}$.

Assume $sw < w$ and $sx < x$:

$$P_{x,w} = qP_{x,sw} + P_{sx,sw} - \sum_{\substack{z \leq sw \\ sz < z}} \mu(z, sw) q^{\frac{l(w)-l(z)}{2}} P_{x,z}.$$

On μ

Q: What is $\deg(P_{x,w})$?

Q: When is $\mu(x, w)$ nonzero?

Q: What values can $\mu(x, w)$ attain?

0-1 conjecture

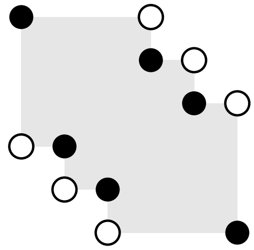
Conjecture: For S_n, $\mu(x, w) \in \{0, 1\}$.

Counterexamples (M):

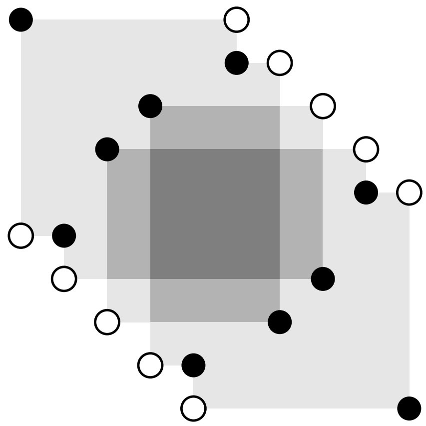
$$x = [6, 5, 2, 1, 10, 9, 4, 3, 14, 13, 8, 7, 12, 11, 16, 15]$$
$$w = [13, 9, 2, 1, 14, 10, 5, 3, 16, 11, 6, 4, 12, 7, 15, 8]$$
$$l(x) = 32, \quad l(w) = 53,$$
$$P_{x,w} = 5q^{10} + 72q^9 + 387q^8 + 1039q^7 + 1610q^6 + 1536q^5 + 931q^4 + 365q^3 + 92q^2 + 14q + 1.$$

$$x = [8, 7, 4, 2, 1, 13, 12, 6, 5, 3, 11, 10, 9, 16, 15, 14]$$
$$w = [13, 11, 7, 2, 1, 16, 12, 8, 4, 3, 14, 9, 5, 15, 10, 6]$$
$$l(x) = 39, \quad l(w) = 60,$$
$$P_{x,w} = 5q^{10} + 56q^9 + 231q^8 + 533q^7 + 776q^6 + 755q^5 + 501q^4 + 226q^3 + 67q^2 + 12q + 1.$$

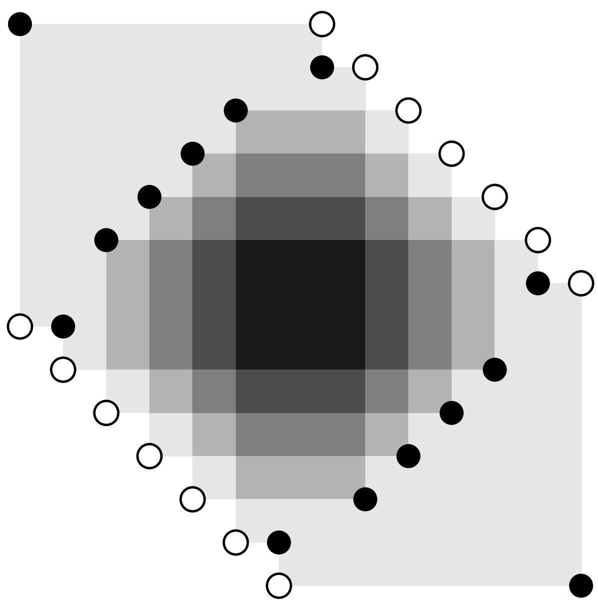
A family (M-W)



$1 + 4q + q^2$



$1 + 16q + 75q^2 + 124q^3 + 58q^4 + 4q^5$



$1 + 36q + 455q^2 + 2564q^3 + 6359q^4 + 7090q^5 + 3462q^6 + 620q^7 + 19q^8$

Sequences

$a_n = 4a_{n-1} + 3a_{n-2}$	0,1,4,19,88,409,1900, 8827,41008,190513,...
Trees of diameter 8	1,4,19,66,219,645, 1813,4802,12265,...
Powers of $\sqrt{19}$ rounded down	1,4,19,82,361,1573, 6859,29897,130321,...

Coefficients of KL polynomials

Q: Is $P_{x,w} \in \mathbb{N}[q]$?

Notation:

- $SL(n)$: $n \times n$ matrices of determinant 1.
- B: subgroup of upper triangular matrices.

Bruhat decomposition: $SL(n) = \amalg_{w \in S_n} BwB$.

Definitions:

- *Flag manifold:* $SL(n)/B = \amalg_{w \in S_n} BwB/B$.
- *Schubert cell:* $X_w^\circ := Bw$.
- *Schubert variety:* $X_w := \overline{X_w^\circ}$.

Facts:

- $X_w^\circ \cong \mathbb{C}^{l(w)}$.
- $X_w = \amalg_v X_v^\circ$.

Intersection Cohomology

Let $\mathrm{IH}^*(X_w)$ denote the (middle) intersection cohomology of the Schubert variety X_w.

Let $\mathrm{IH}^*_x(X_w)$ denote the corresponding stalk at a point $x \in X_w$.

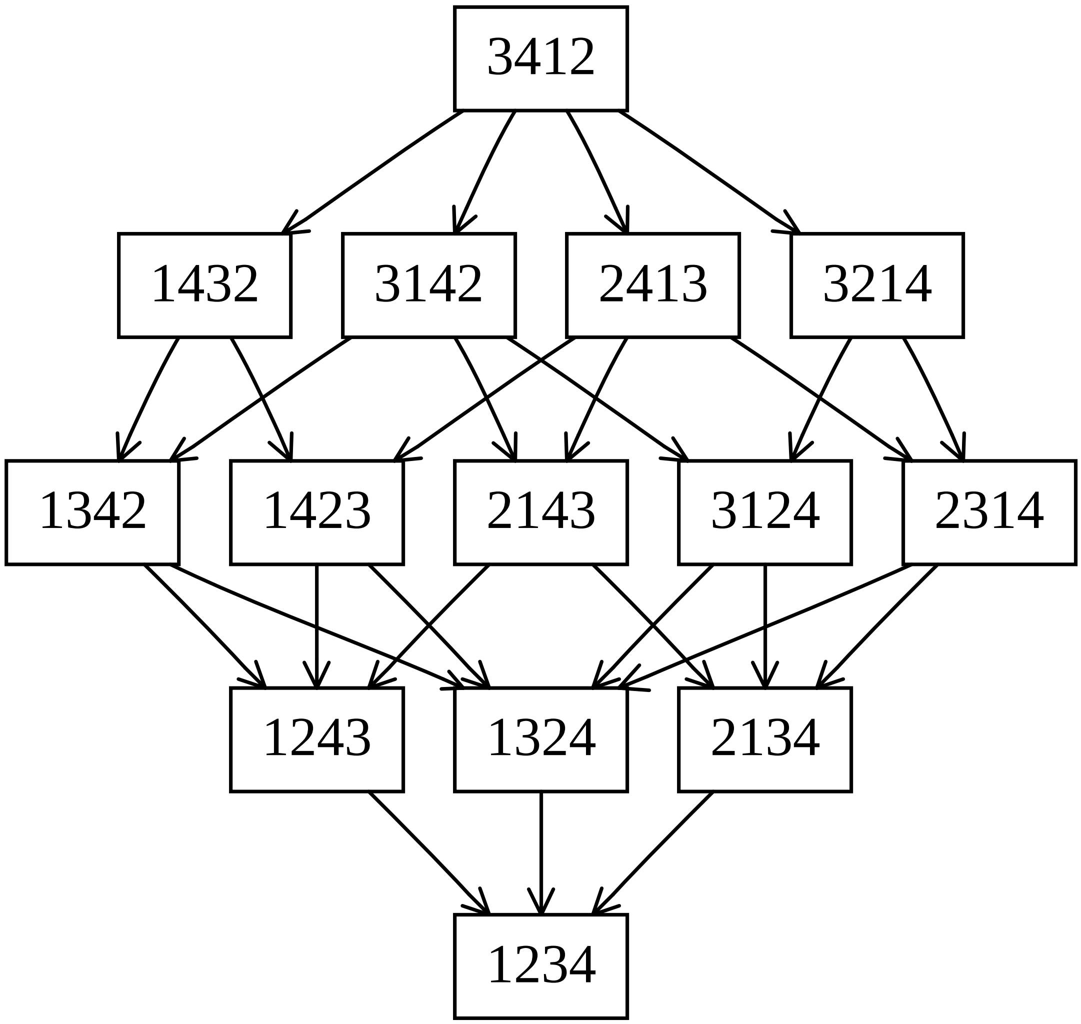
Theorem 2 (K-L). *For W a Weyl group,*

$$\sum_{x \leq w} q^{l(x)} P_{x,w}(q) = \sum_i \dim(\mathrm{IH}^{2i}(X_w))q^i,$$

$$P_{x,w}(q) = \sum_i \dim(\mathrm{IH}^{2i}_x(X_w))q^i.$$

NB: The second equality implies $P_{x,w}(q) \in \mathbb{N}[q]$ when W is a Weyl group.

Duality



$$\sum_{x \leq w} q^{l(x)} P_{x,w}(q) = \sum_i \dim(\mathrm{IH}^{2i}(X_w)) q^i.$$

Coefficients of KL polynomials

Q: Is $P_{x,w} - P_{y,w} \in \mathbb{N}[q]$ for $x \leq y$?

Q: Can any polynomial subject to the above conditions be realized as a KL polynomial?

Q: What are the coefficients of $P_{x,w}$ counting?

Q: When does $P_{x,w} = 1$?

Singularities

X_w is **singular at the point** p iff

$$\dim(T_p(X_w)) > \dim(X_w) = l(w).$$

Theorem 3 (K-L,P). *If W is a "simply-laced" Coxeter group, then $P_{x,w}(q) = 1$ iff x is a non-singular point of X_w.*

Q: Is there a combinatorial characterization of when $P_{x,w}(q) = 1$?

Pattern avoidance

$w \in S_n$ is **321-avoiding** if there do not exist $i < j < k$ with $w(i) > w(j) > w(k)$.

Theorem 4 (L-S). *For $W = S_n$, $P_{e,w} = 1$ if and only if w is 3412- and 4231-avoiding.*

Example:
$w = [6, 4, 1, 7, 2, 5, 3]$ is not 3412- or 4231-avoiding.

Theorem 5 (B-W,C,K-L-R,M). *For fixed $w \in S_n$, the maximal elements $x \leq w$ such that $P_{x,w} \neq 1$ can be characterized in terms of patterns.*

Q: What does $P_{x,w}(1)$ count?

Lie algebras

Let $\mathfrak{g}$ be a **finite-dimensional complex semisimple Lie algebra**:

- is a finite-dimensional complex vector space,
- has no nonzero abelian ideals,
- has a skew-symmetric bilinear form $[\cdot, \cdot]$, and
- $[\cdot, \cdot]$ satisfies the Jacobi identity.

Example:
$\mathfrak{sl}_n =$ traceless $n \times n$ matrices.
$[X, Y] = XY - YX$.

Cartan subalgebras

Q: What do representations of $\mathfrak{g}$ look like?

Let $\mathfrak{h}$ be a **Cartan subalgebra** of $\mathfrak{g}$:

- a commutative subalgebra,
- that is maximal,
- and simultaneously diagonalizable

Example: For $\mathfrak{sl}_n$, $\mathfrak{h} \leftrightarrow$ traceless diagonal $n \times n$ matrices.

Weights & Roots

Let $\mathfrak{h} = \langle H_1, \dots, H_r \rangle$.

Example: For $\mathfrak{sl}_n$, $H_i = E_{ii} - E_{i+1,i+1}$.

$(m_1, \dots, m_r) \in \mathbb{C}^r \setminus 0$ is a **weight** of π if there exists a nonzero $v \in V$ such that $\pi(H_i)v = m_i v$ for all i.

Fact: Weights are in $\mathbb{Z}^r$.

A **root** is a weight of the adjoint representation

$$ad_X(Y) = [X, Y].$$

$\mathfrak{sl}_3$

Generators of $\mathfrak{sl}_3$:

$$H_1 = E_{1,1} - E_{2,2},$$
$$H_2 = E_{2,2} - E_{3,3},$$
$$X_1 = Y_1^t = E_{1,2},$$
$$X_2 = Y_2^t = E_{2,3},$$
$$X_3 = Y_3^t = E_{1,3}.$$

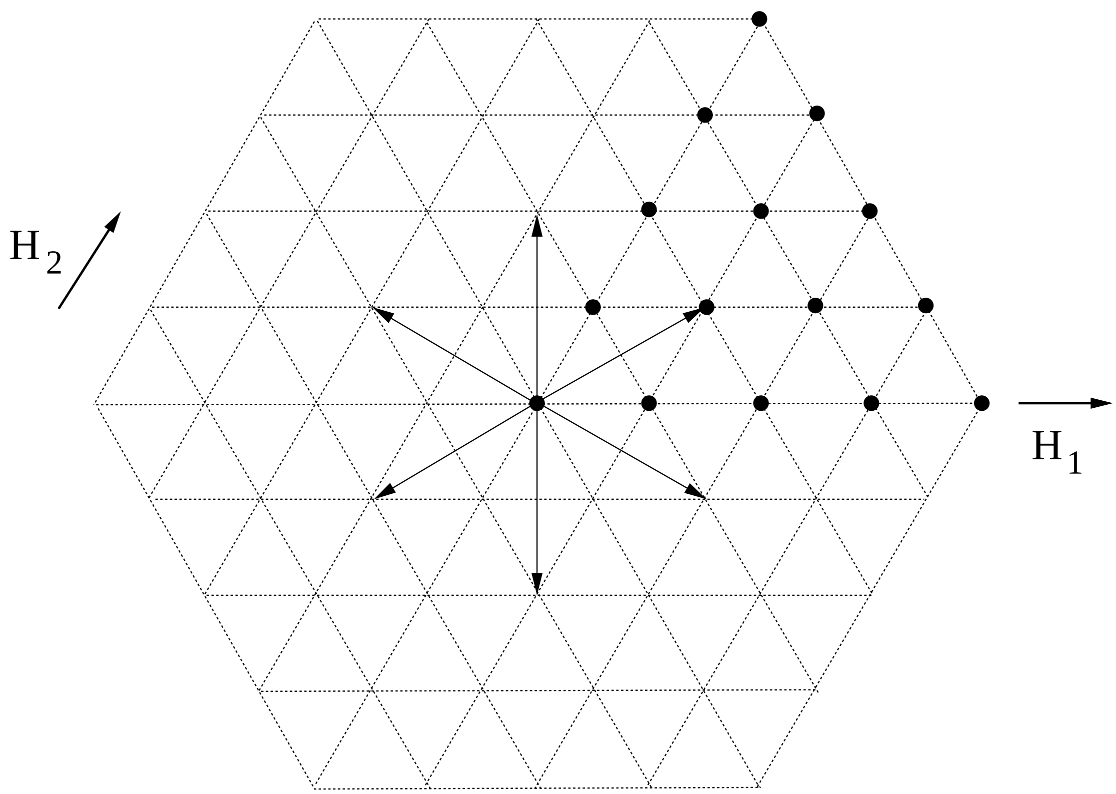
Adjoint action of H_1 and H_2:

$$[H_1, X_1] = 2\ X_1, \quad [H_1, Y_1] = -2Y_1,$$
$$[H_2, X_1] = -X_1, \quad [H_2, Y_1] = \quad Y_1,$$

$$[H_1, X_2] = -X_2, \quad [H_1, Y_2] = \quad Y_2,$$
$$[H_2, X_2] = 2\ X_2, \quad [H_2, Y_2] = -2Y_2,$$

$$[H_1, X_3] = \quad X_3, \quad [H_1, Y_3] = \ -Y_3,$$
$$[H_2, X_3] = \quad X_3, \quad [H_2, Y_3] = \ -Y_3.$$

Weyl group



The **Weyl group** of $\mathfrak{g}$ is the symmetry group of the roots. It is a Coxeter group. For an appropriate inner product on $\mathfrak{h}$, it is generated by reflections.

Theorem of the highest weight

Fact: There is a natural partial order $\preceq$ on weights.

A **highest weight module**, V_λ, is a representation of $\mathfrak{g}$ with weight λ such that if ν is a weight of V_λ, then $\nu \preceq \lambda$.

Theorem 6. *Every finite-dimensional irreducible representation of $\mathfrak{g}$ is a highest weight module for some* ***dominant integral weight****.*

More facts:

1. A **Verma module**, M_λ, is a universal highest weight module.
2. If λ is dominant integral, then M_λ has a unique finite-dimensional irreducible quotient, L_λ.

The Kazhdan-Lusztig conjecture

Setup:

1. Pick a dominant integral weight, λ_0.
2. Define a shifted action of W on weights "$x \cdot \lambda$".
3. Let $[M_{x\cdot\lambda_0} : L_\nu]$ denote the multiplicity of L_ν in $M_{x\cdot\lambda_0}$.

Theorem 7 (V,B-G-G,vdH). $[M_{x\cdot\lambda_0} : L_\nu] \neq 0$ *iff* $\nu = y \cdot \lambda_0$ *for some* $y \geq x$.

Theorem 8 (B-G-G). $m(x,y) = [M_{x\cdot\lambda_0} : L_{y\cdot\lambda_0}]$ *is independent of* λ_0.

Theorem 9 (Conj. of K-L; B-B,B-K).

$$m(x,y) = P_{x,y}(1).$$