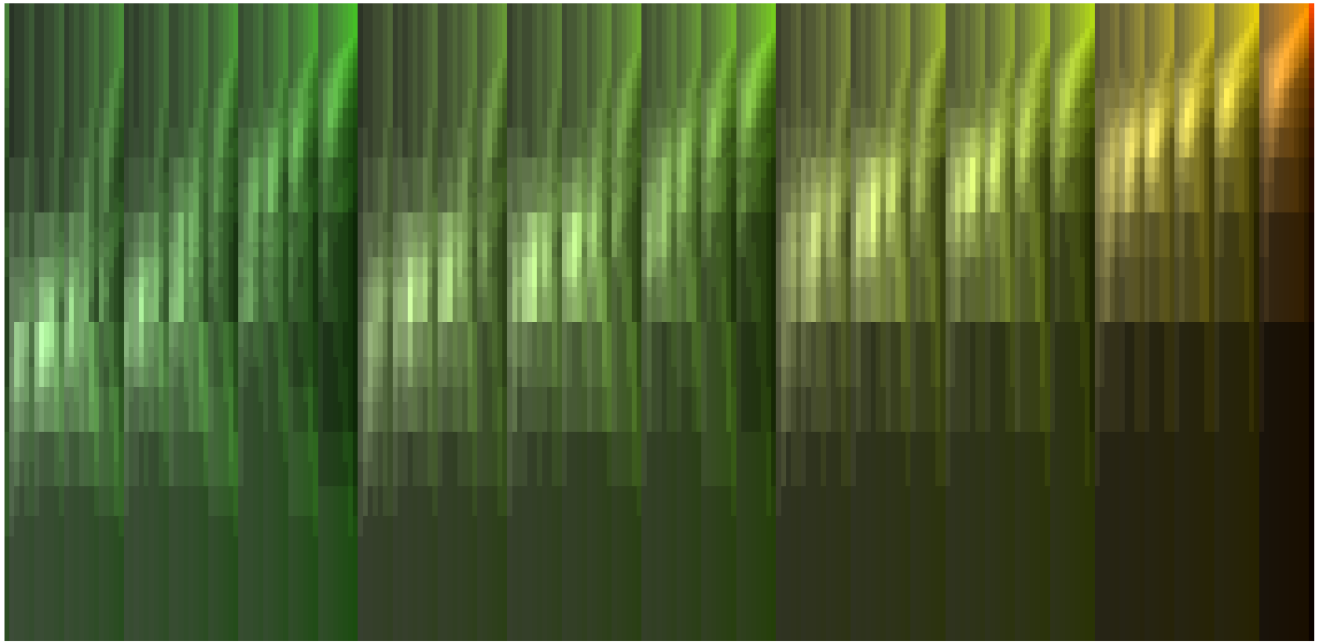
Statistics in combinatorics

Greg Warrington

The University of Vermont



MathFest: August 8, 2009

	Objects	Concepts/ Techniques	Other
Geometry	Shapes, manifolds		Euclid, picture
Topology	Shapes	Topological equivalence Open/closed	rubber sheet
Number theory	Primes	Quadratic reciprocity Euler phi function	Cryptography, a b c d
Algebra	Groups	Variables	Structure, lower-level college
Analysis	Functions, integrals	ε/δ, distance, limits, measure	Elegance
Combina-torics	Graphs	Counting	Variations

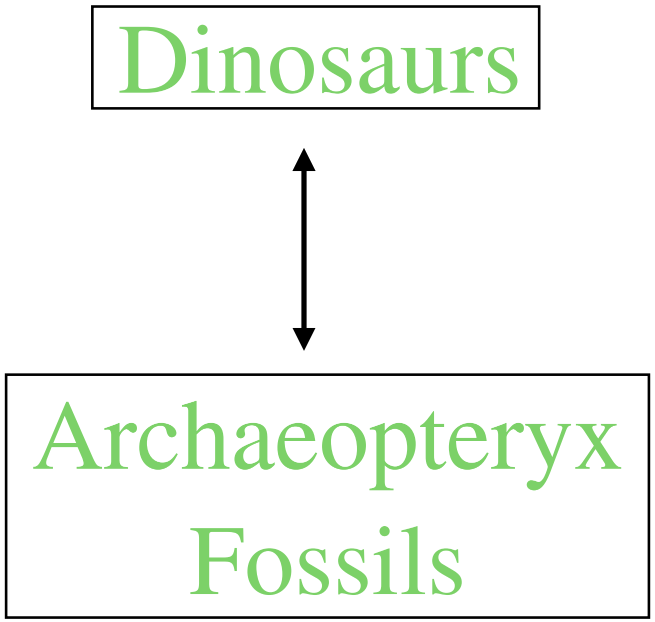
Survey of Project NExT Fellows, n=11

combinatorics is
intimately connected
to other areas of
mathematics

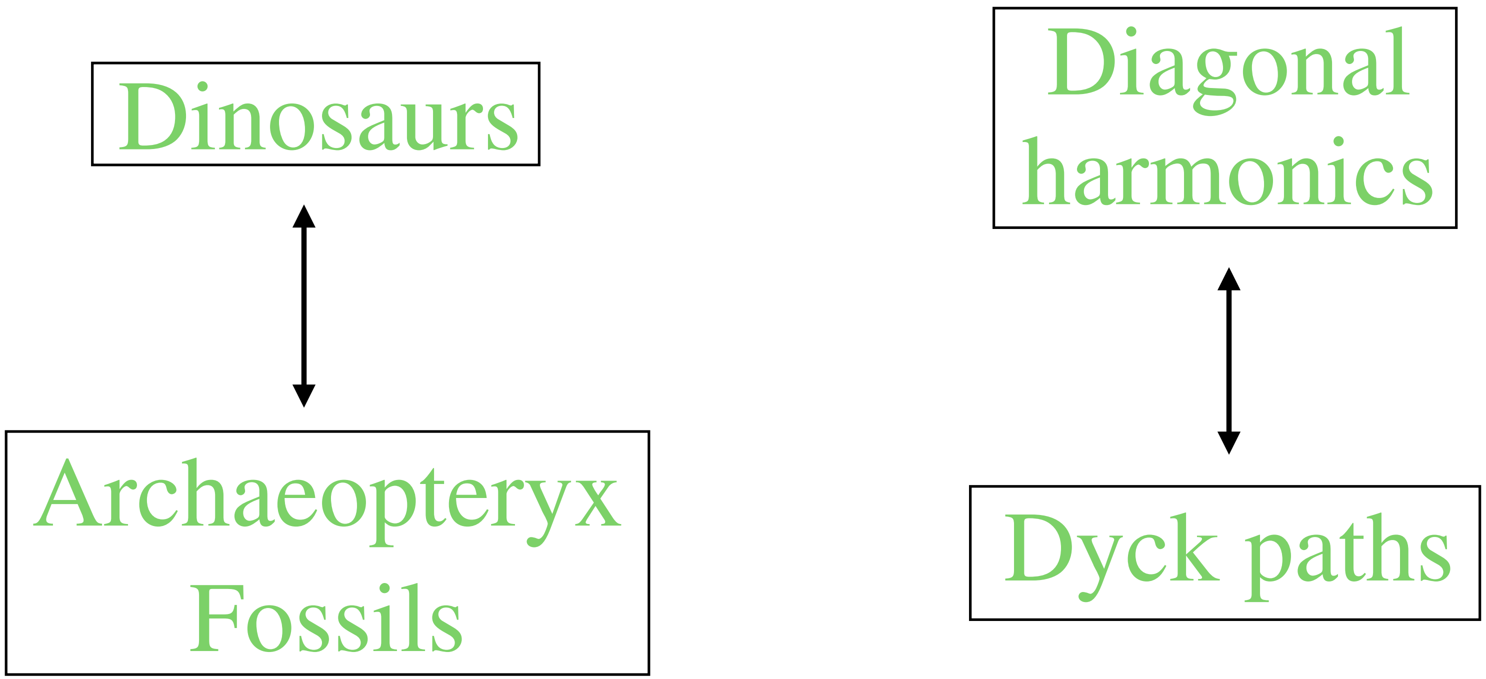
through

discrete objects

Theory of Evolution



Theory of Evolution	Representation Theory



$$\boldsymbol{C_n(q,t) = \sum_{\mu \vdash n} \frac{T_\mu^2 M \Pi_\mu B_\mu}{w_\mu}}$$

$$\boldsymbol{C_n(q,t) = \sum_{\mu \vdash n} \frac{T_\mu^2 M \Pi_\mu B_\mu}{w_\mu}}$$

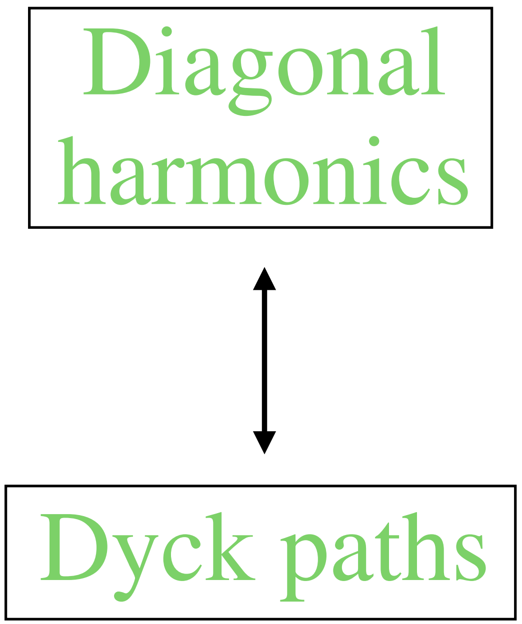
Polynomial?

$$\boxed{\boldsymbol{C_3(q,t) = q^3 + tq^2 + tq + t^2q + t^3}}$$

Nonnegative?

Symmetric?

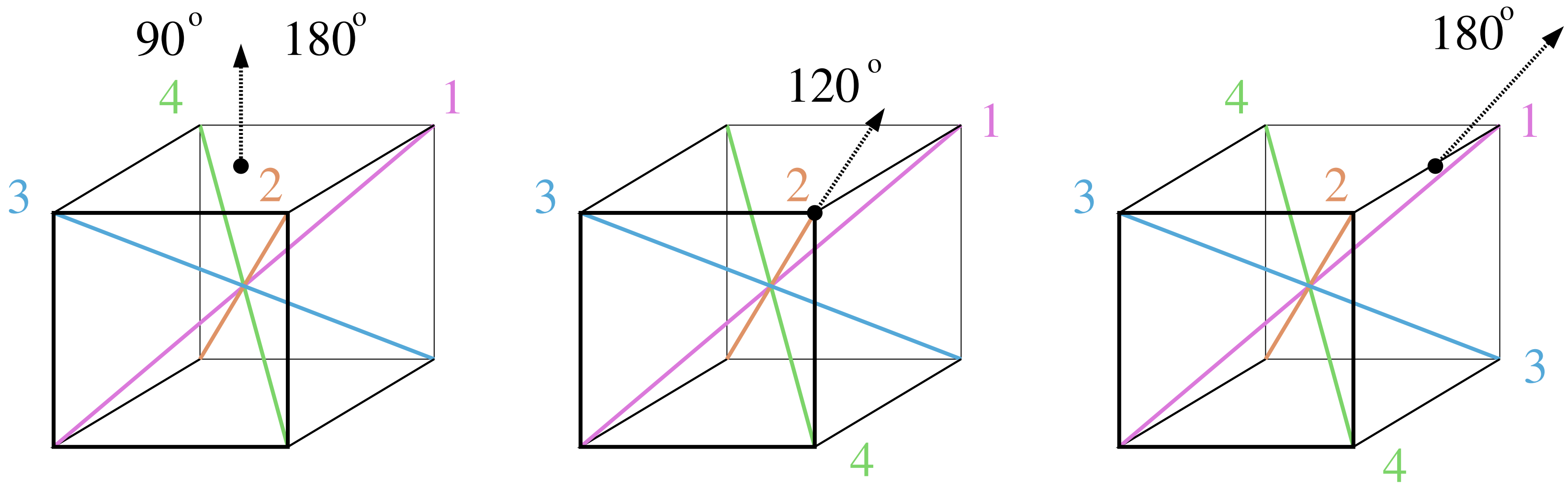
Representation Theory

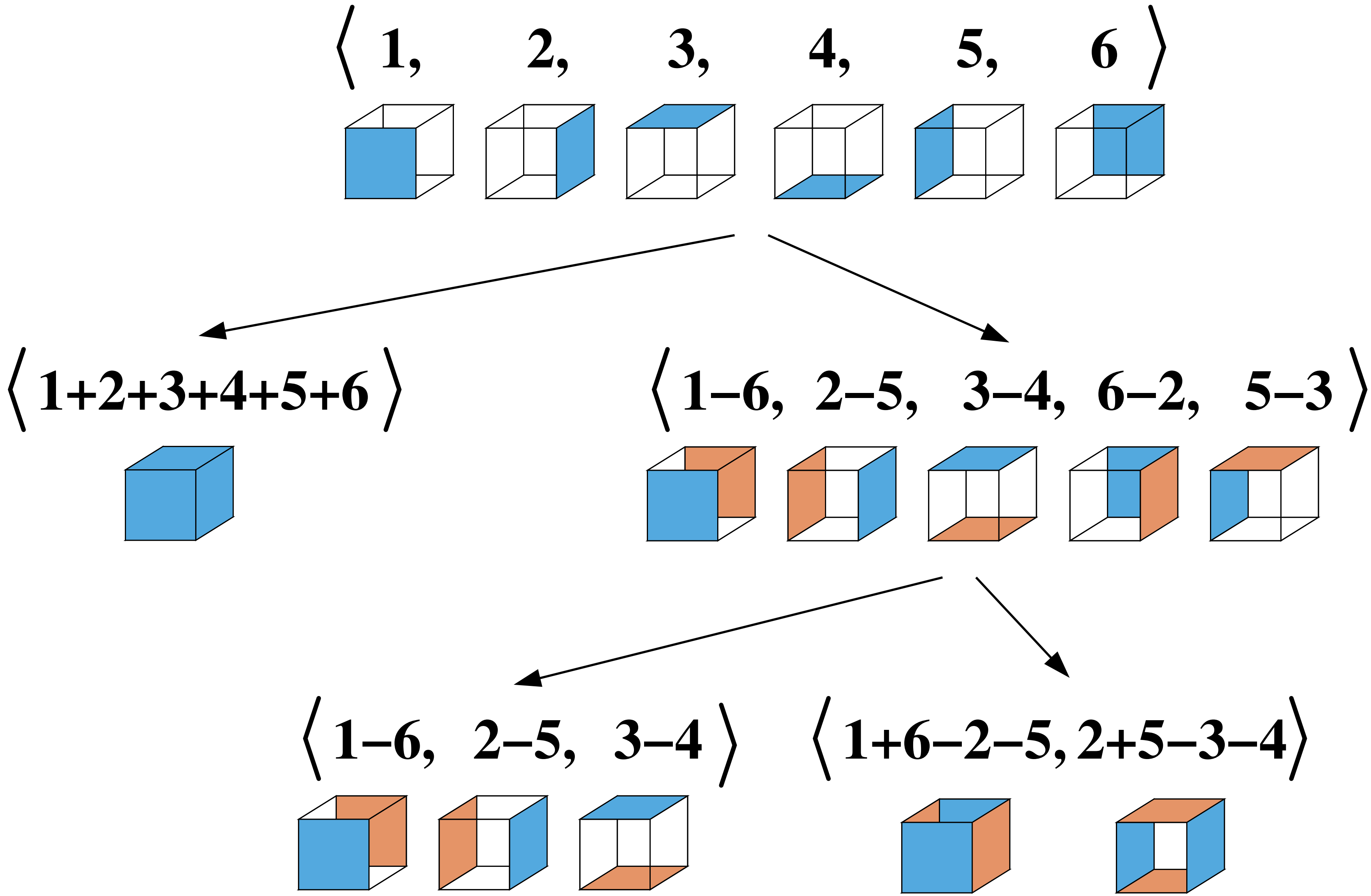


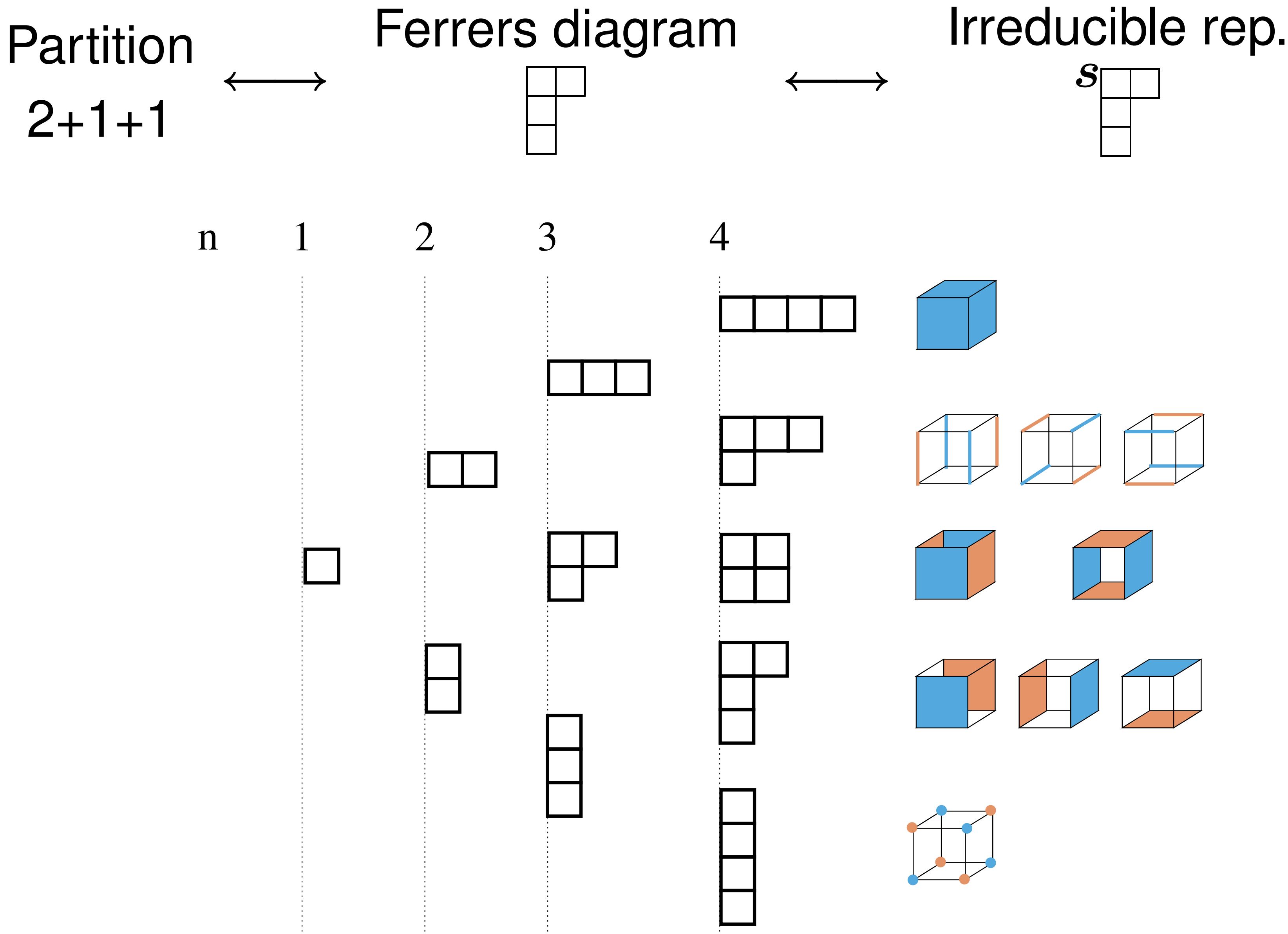
$$\boldsymbol{S_4} = \{\text{Permutations of } [\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}]\}$$

$$= \langle(\mathbf{1}, \mathbf{2}),\ (\mathbf{2}, \mathbf{3}),\ (\mathbf{3}, \mathbf{4})\rangle$$

$$= \text{Rotation group of cube.}$$




⟨ 1, 2, 3, 4, 5, 6 ⟩
⟨ 1+2+3+4+5+6 ⟩
⟨ 1−6, 2−5, 3−4, 6−2, 5−3 ⟩
⟨ 1−6, 2−5, 3−4 ⟩
⟨ 1+6−2−5, 2+5−3−4 ⟩


Partition
2+1+1
Ferrers diagram
Irreducible rep.
s
n
1
2
3
4

Let $R = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$.

R carries a representation of S_n:

$$\sigma x_i = x_{\sigma_i} \qquad \sigma y_i = y_{\sigma_i}.$$

Diagonal harmonics module

$$\mathrm{DH}_n = \left\{ f \in R : \sum_{i=1}^{n} \partial_{x_i}^{h} \partial_{y_i}^{k} f = 0, \forall\, h + k > 0 \right\}$$

$$\mathbf{DH}_2 = \left\{ f \in R : \begin{array}{l} (\partial_{x_1} + \partial_{x_2}) f = 0 \\ (\partial_{y_1} + \partial_{y_2}) f = 0 \\ \qquad \vdots \end{array} \right\}.$$

$$\mathbf{DH}_2 = \langle 1 \rangle + \langle x_2 - x_1 \rangle + \langle y_2 - y_1 \rangle$$

$s_{\square\square}$	$s_{\substack{\square\\\square}}$	$s_{\substack{\square\\\square}}$
Trivial	Sign	Sign

DH_n is bigraded by total degree

$$\mathbf{DH}_n = \oplus_{i,j} \boldsymbol{V}^{i,j}$$

Example: $\boldsymbol{x_1^3\, x_2\, x_3^4\, y_2\, y_4 \in V^{8,2}}$.

Hilbert series

$$\boldsymbol{\mathcal{H}}(\mathbf{DH}_n) = \sum_{i,j} \mathbf{dim}(\boldsymbol{V}^{i,j})\boldsymbol{q}^i \boldsymbol{t}^j$$

$$\mathrm{DH}_3 = \underbrace{\langle 1 \rangle}_{s_{(3)}} + \underbrace{\langle x_2 - x_1, x_3 - x_2 \rangle}_{s_{(2,1)}} + \cdots + \underbrace{\langle y_3(x_2 - x_1) + x_3(y_1 - y_2) + x_1 y_2 - x_2 y_1 \rangle}_{s_{(1,1,1)}} + \cdots$$

$$\mathcal{H}(\mathrm{DH}_3|_{s_{(3)}}) = 1,$$

$$\mathcal{H}(\mathrm{DH}_3|_{s_{(2,1)}}) = t + q + q^2 + tq + t^2,$$

$$\mathcal{H}(\mathrm{DH}_3|_{s_{(1,1,1)}}) = tq + t^3 + t^2 q + tq^3 + q^3.$$

Write $C_n(q,t) = \mathcal{H}(\mathrm{DH}_n\,|_{s_1 n})$.

Write $\boldsymbol{C_n(q,t) = \mathcal{H}(\mathbf{DH}_n|_{s_1 n})}$.

---

2 $\qquad \boldsymbol{C_2(q,t) = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}}$

5 $\qquad \boldsymbol{C_3(q,t) = \begin{bmatrix} & & & 1 \\ & 1 & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}}$

14 $\qquad \boldsymbol{C_4(q,t) = \begin{bmatrix} & & & & & & 1 \\ & & & 1 & 1 & 1 & \\ & & 1 & 1 & 1 & & \\ & 1 & 1 & 1 & & & \\ & 1 & 1 & & & & \\ & 1 & & & & & \\ 1 & & & & & & \end{bmatrix}}$

$$\boldsymbol{c_n = 1, 1, 2, 5, 14, 42, 132, 429, \ldots}$$

$$\boldsymbol{= c_0 c_{n-1} + c_1 c_{n-2} + \cdots + c_{n-1} c_0}$$

$$\boldsymbol{= \frac{1}{n+1}\binom{2n}{n}.}$$

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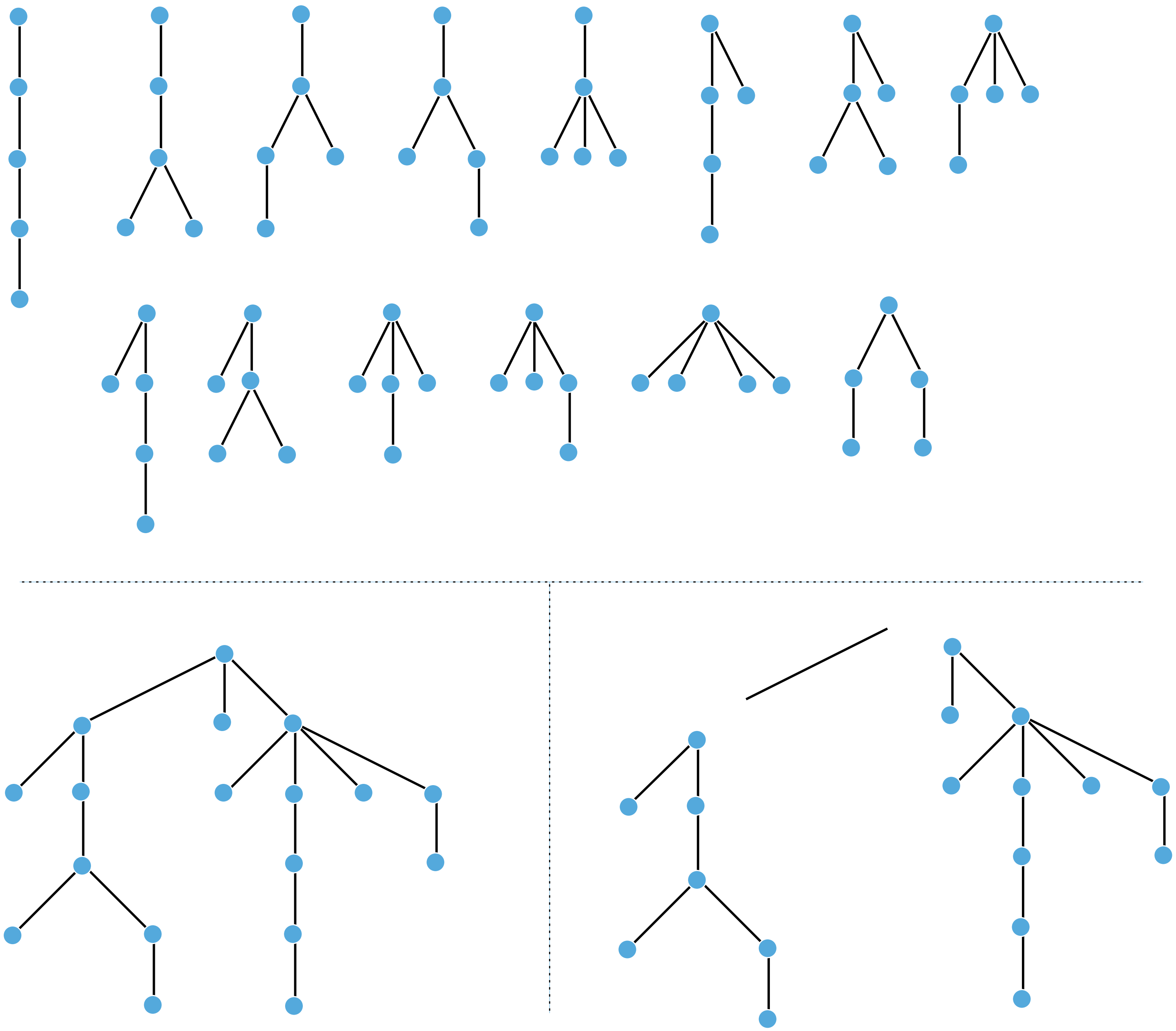
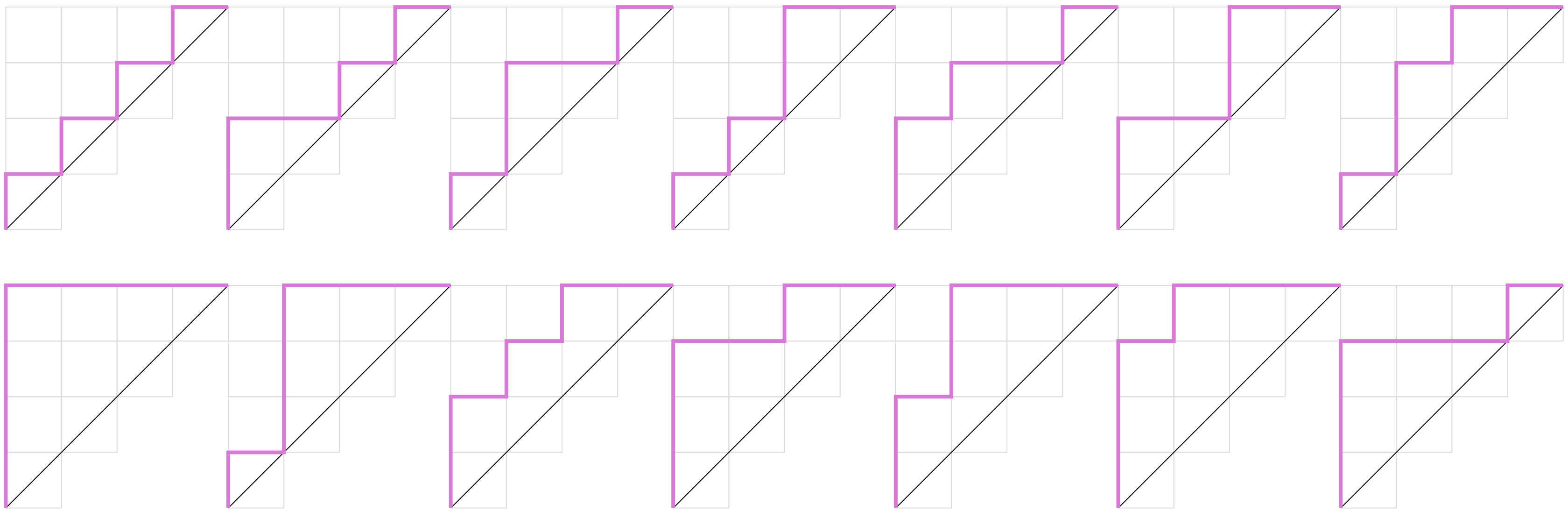
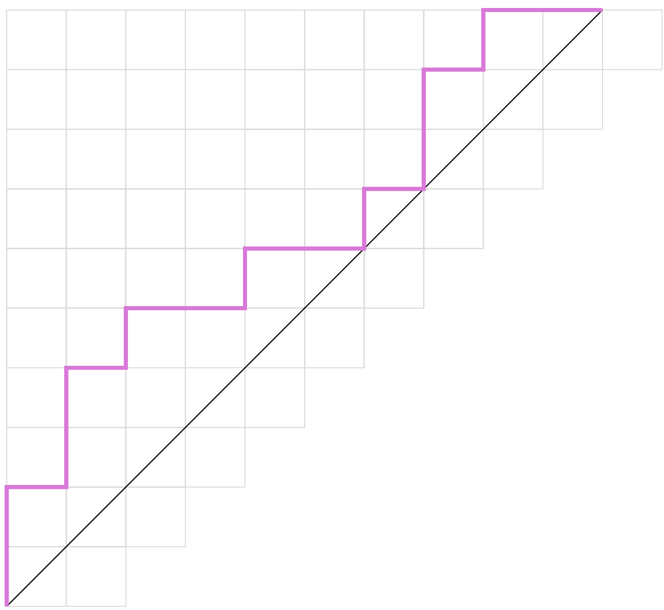
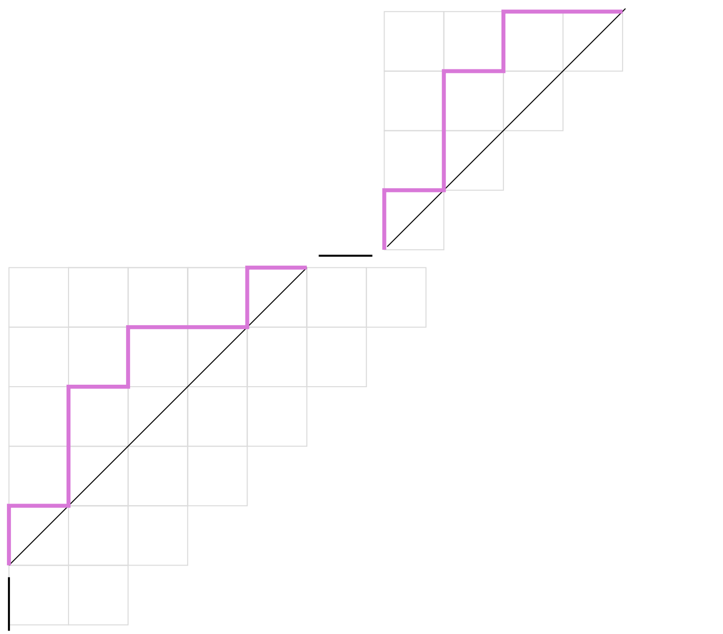
http://www.research.att.com/~njas/sequences/



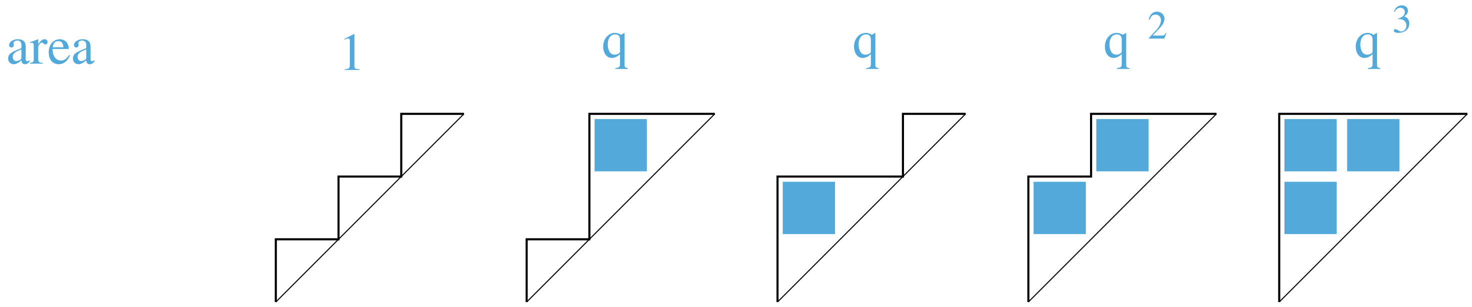






$$\boldsymbol{C_3(q,1) = 1 \ + \ q + \ q \ + \ q^2 \ + \ q^3}$$

$$C_3(q,1) = 1 + q + q + q^2 + q^3$$



Wanted: A "$\mathbf{tstat}$" such that

$$\boldsymbol{C_n(q,t) = \sum_{D \in \mathrm{D}_n} q^{\mathbf{area}(D)} t^{\mathbf{tstat}(D)}}.$$

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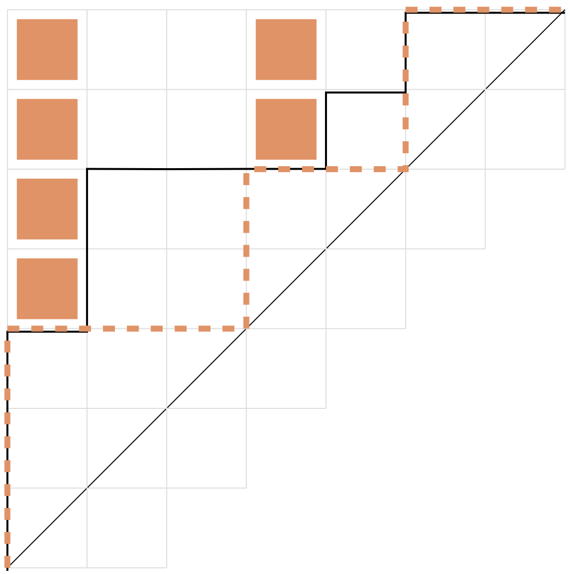
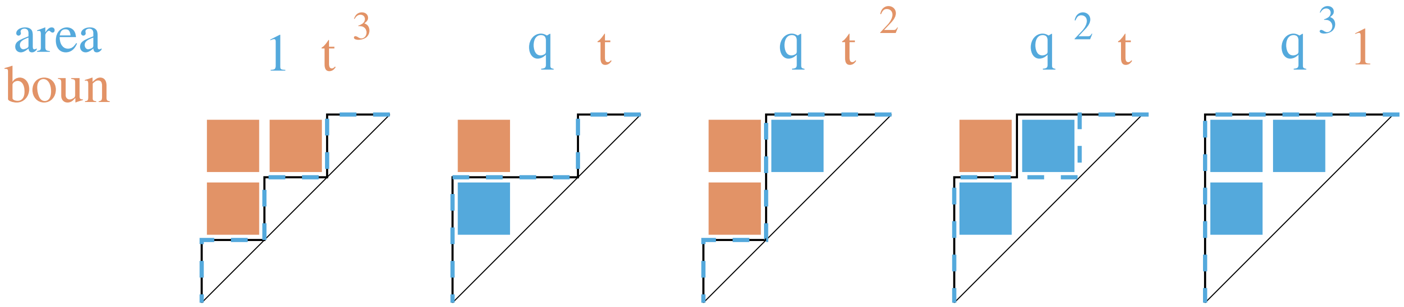
$$\boldsymbol{C_n(q,t) = \sum_{D \in \mathrm{D}_n} q^{\mathrm{area}(D)} t^{\mathrm{tstat}(D)}}.$$

Haglund proposed "bounce" for $\mathbf{tstat}$.



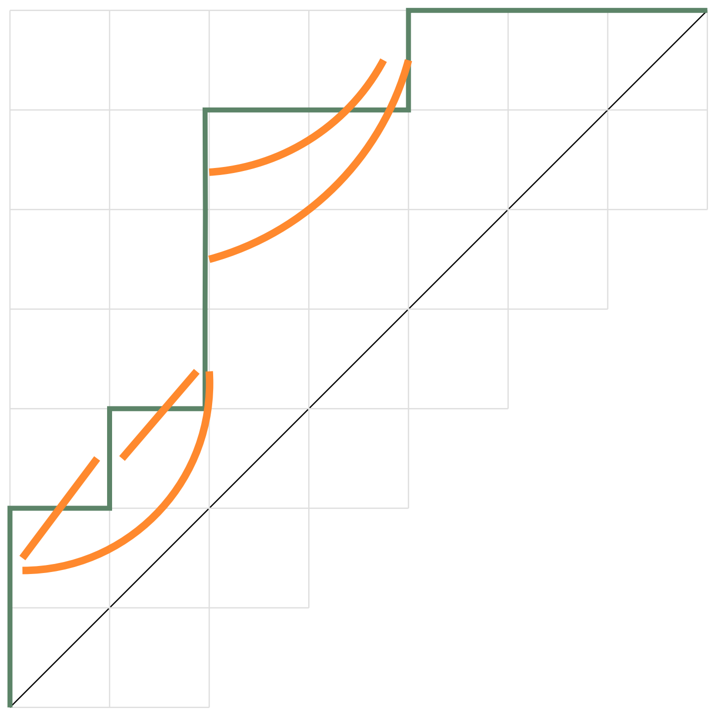
$$\mathbf{boun = 4 + 2 + 0}$$

$$C_3(q,t) = t^3 + qt + qt^2 + q^2t + q^3$$



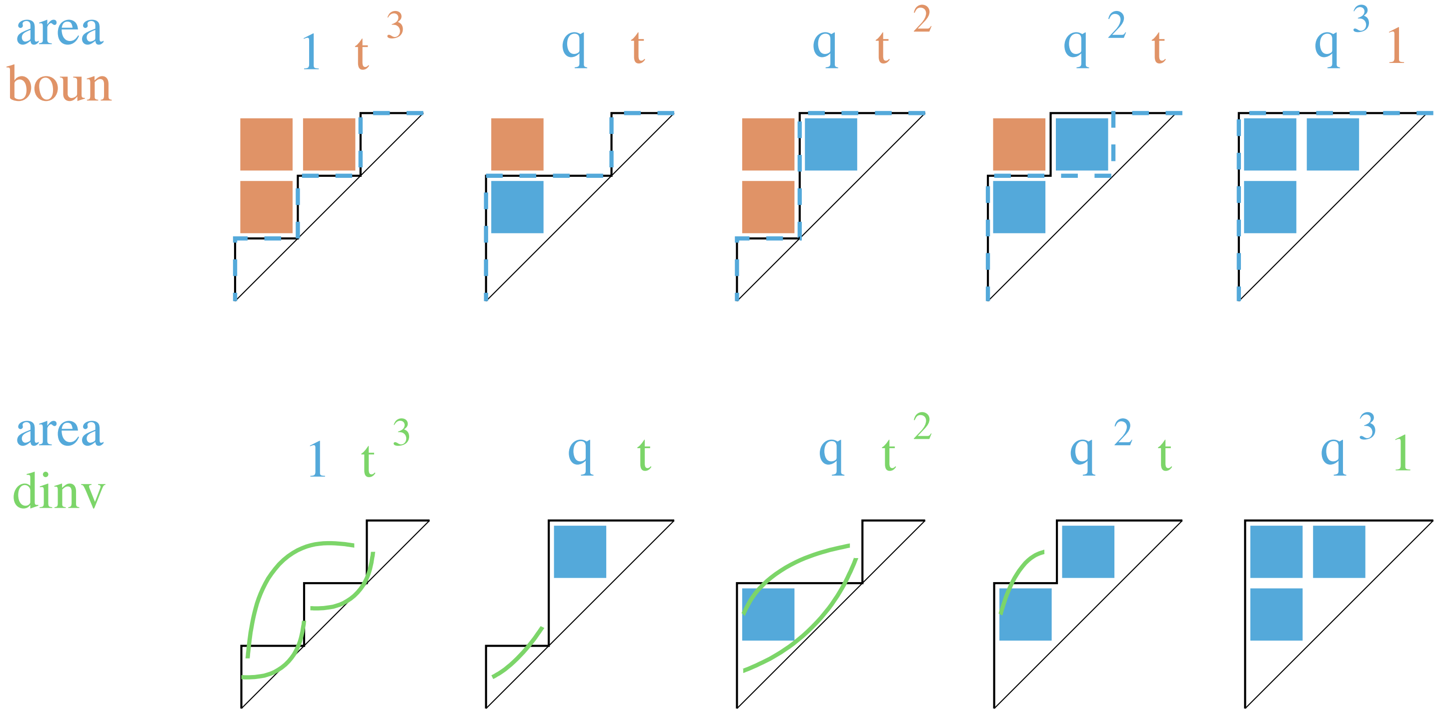
Conj: Haglund ’00; Thm: Garsia-Haglund ’01

Haiman proposed “dinv” for $\mathbf{tstat}$.



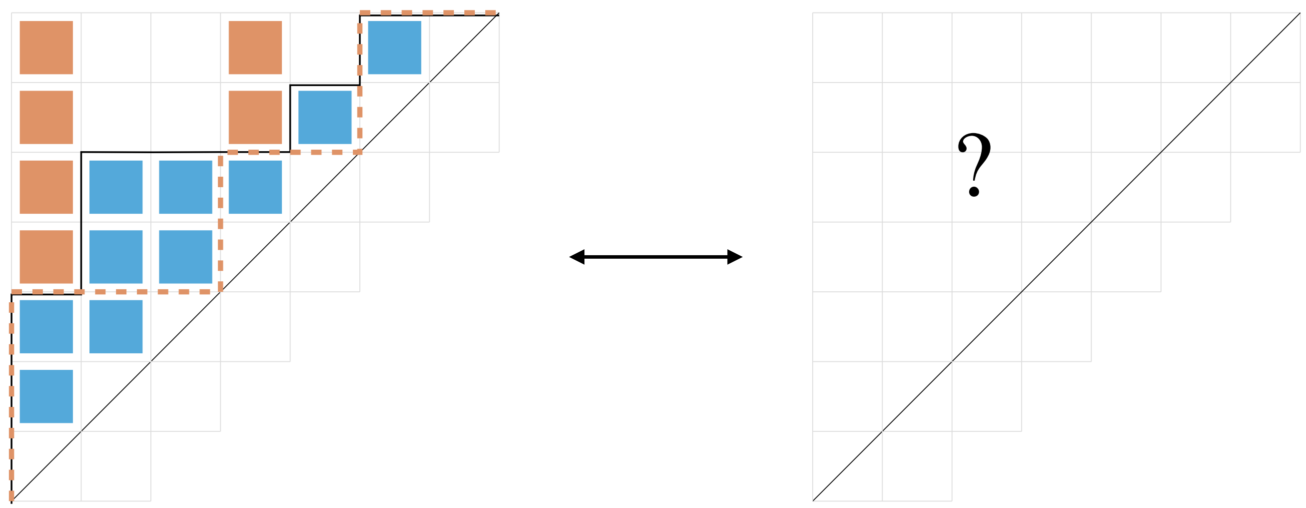
$$\mathbf{dinv} = \mathbf{5}$$

$$C_3(q,t) = t^3 + qt + qt^2 + q^2t + q^3$$



Still no combinatorial proof that

$$C_n(q,t) = C_n(t,q).$$



$$\mathbf{area} = 10 \qquad \mathbf{area} = 6$$

$$\mathbf{boun} = 6 \qquad \mathbf{boun} = 10$$

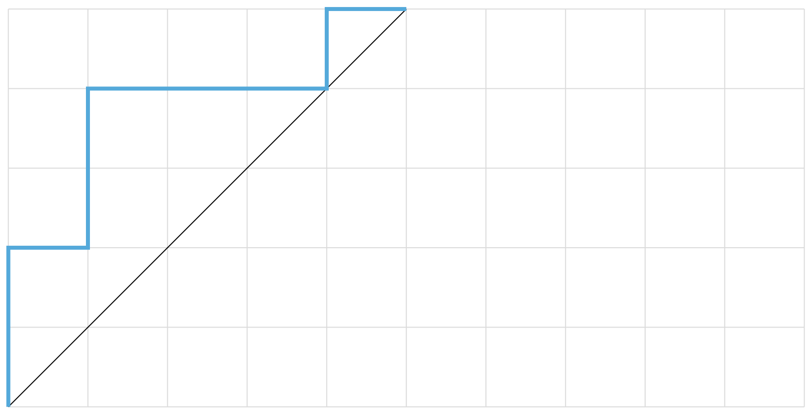
$\mathbf{\Lambda}$: the ring of symmetric functions

Bases: $\boldsymbol{s_\mu, m_\mu, e_\mu, h_\mu, p_\mu, \tilde{H}_\mu, \ldots}$.

Linear operator $\boldsymbol{\nabla}$: $\mathbf{\Lambda} \longrightarrow \mathbf{\Lambda}$.

Goal: Find matrices for $\boldsymbol{\nabla}$.

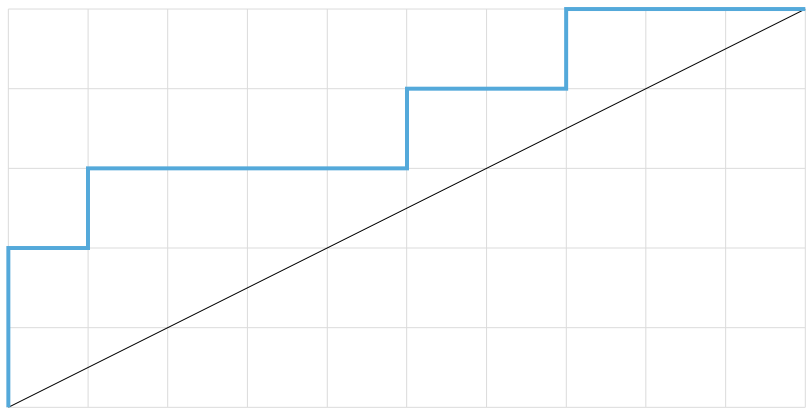
Algebraic Object	Combinatorial Model	Conjecture	Proof
$\nabla(s_{1^n})\|_{s_{1^n}}$	Dyck	Hg	G-Hg



C=Can, E=Egge, G=Garsia, Hg=Haglund, H=Haiman, K=Kilpatrick,

Kr=Kremer, Le=Lenart, L=Loehr, R=Remmel, U=Ulyanov, W=W

Algebraic Object	Combinatorial Model	Conjecture	Proof
$\nabla^m(s_{1^n})\|_{s_{1^n}}$	m-Dyck	L	



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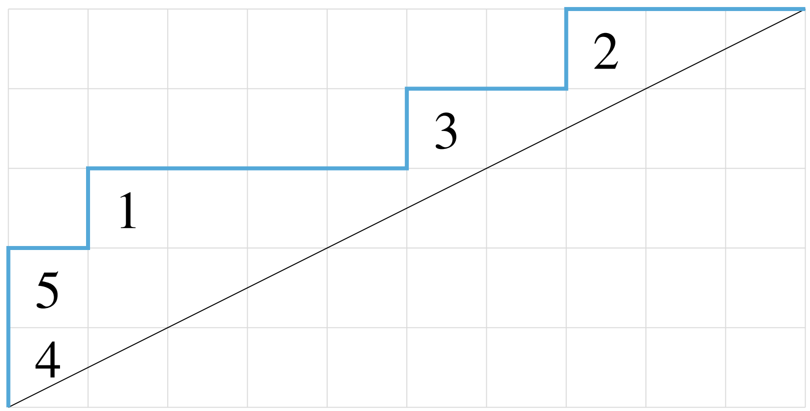
Algebraic Object	Combinatorial Model	Conjecture	Proof
$\nabla^m(s_{1^n})\|_{m_{1^n}}$	labeled m-Dyck	L-R	



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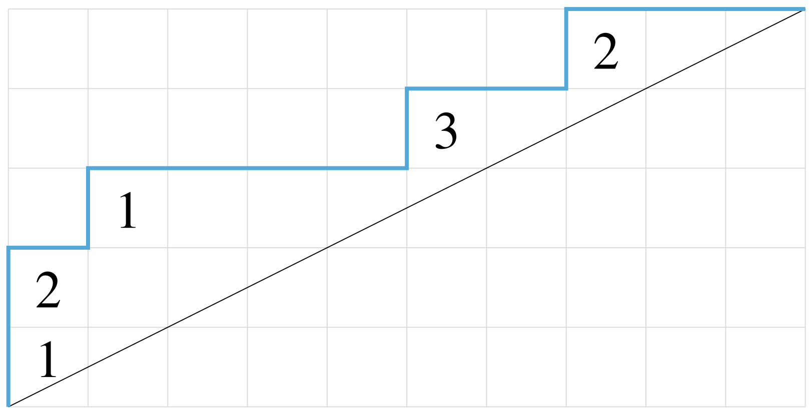
Algebraic Object	Combinatorial Model	Conjecture	Proof
$\nabla^m(s_{1^n})\|_{m_\lambda}$	labeled m-Dyck	Hg-H-L-R-U	



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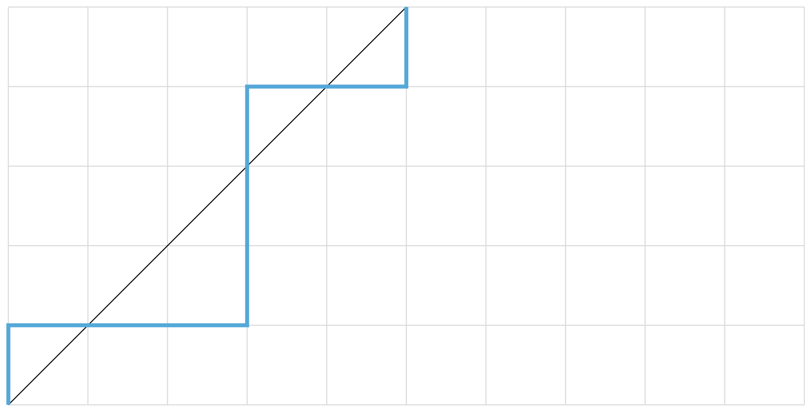
Algebraic Object	Combinatorial Model	Conjecture	Proof
$\nabla(p_n)\|_{s_{1^n}}$	square	L-W	C-L



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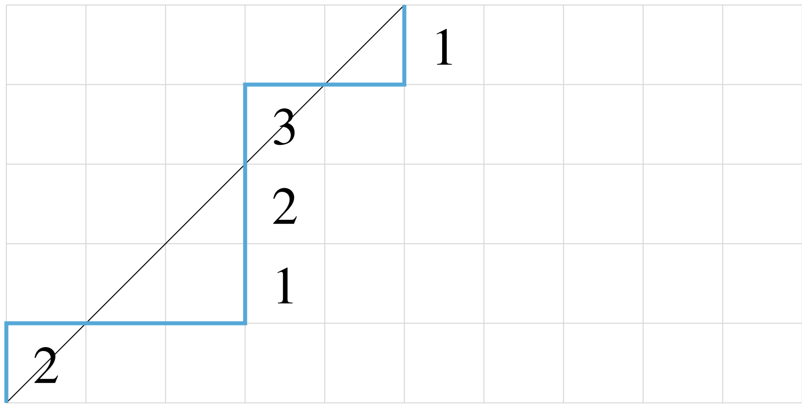
Algebraic Object	Combinatorial Model	Conjecture	Proof
$\nabla(p_n)\|_{m_\lambda}$	labeled square	L-W	



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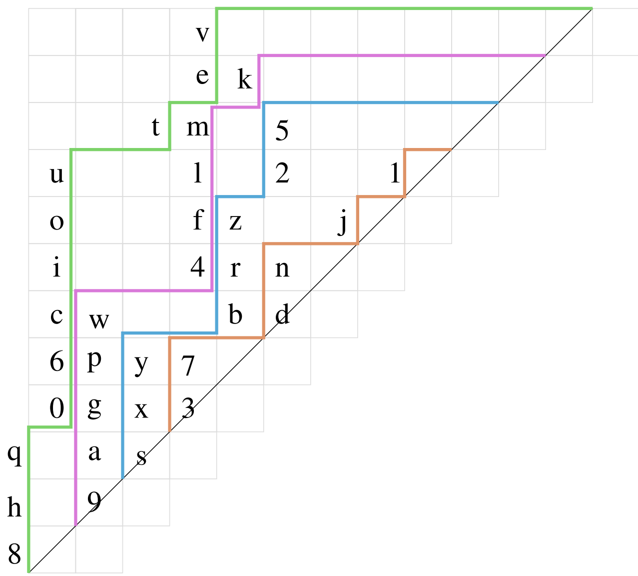
Algebraic Object	Combinatorial Model	Conjecture	Proof
$\nabla^m(s_\mu)\|_{m_\lambda}$	{nest,label}ed m-Dyck	L-W	



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Algebraic Object	Combinatorial Model	Conjecture	Proof
$\nabla(s_{1^n})\|_{m_{1^n}}$	q,t-parking functions	Hg-L	
$\nabla_{q=1}(s_{\mu/\nu})\|_{s_\lambda}$	digraphs	Le	Le
$\langle\nabla(s_{1^n}), e_d h_{n-d}\rangle$	q,t-Schröder paths	E-Hg-K-Kr	Hg
$\tilde{H}_\mu\|_{m_\mu}$	fillings of $\mathcal{F}(\mu)$	Hg	Hg-H-L

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Theorem The following uniquely determine a family $\tilde{H}_\mu(X; q, t)$ of symmetric functions:

- $\tilde{H}_\mu[X(q-1); q, t] = \sum_{\rho \leq \mu'} c_{\rho,\mu}(q,t) m_\rho(X)$,
- $\tilde{H}_\mu[X(t-1); q, t] = \sum_{\rho \leq \mu} d_{\rho,\mu}(q,t) m_\rho(X)$,
- $\tilde{H}_\mu(X; q, t)|_{x_1^n} = 1$.

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versus $\tilde{H}_\mu = \sum$ [Filled Ferrers diagrams with "maj" and "inv" statistics]

$$\tilde{H}_{32} = \cdots + q^{\mathrm{maj}\left(\begin{array}{|c|c|c|}\hline 3 & 1 & 2 \\ \hline 1 & 2 & \\ \cline{1-2}\end{array}\right)} t^{\mathrm{inv}\left(\begin{array}{|c|c|c|}\hline 3 & 1 & 2 \\ \hline 1 & 2 & \\ \cline{1-2}\end{array}\right)} m_{221} + \cdots$$

Why are $\boldsymbol{C_n(q,t)}$ symmetric?

Prove validity of various models

Explain similarity of models

Expand $\boldsymbol{\nabla(a_\mu)|_{b_\mu}}$

“The Waiting Room,” Hermann Dyck (Walther’s father)