Combinatorial aspects of $\nabla(s_\lambda)$

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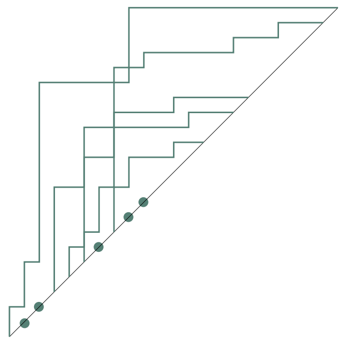

Place: CRM

Date: My wife's birthday, 2007

Outline

- Big example
- Symmetry
- Proof of $\boldsymbol{\nabla}(s_\lambda)$ formula*

* when $q = 1$

The ∇ operator

$$\nabla(\widetilde{H}_\mu) = q^{n(\mu')} t^{n(\mu)} \widetilde{H}_\mu$$

$$\nabla(b_\mu) = ? \text{ for } b \in \{s, h, m, e, p\}$$

The ∇ operator

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$$\nabla(b_\mu) = ? \text{ for } b \in \{s, h, m, e, p\}$$

However,

$$\nabla(e_n) \leftrightarrow \text{ "Labeled Dyck paths"}$$

$$\nabla(p_n) \leftrightarrow \text{ "Labeled square paths"}$$

Main Result

Conjecture (NL,GW)**.** For any partition $\boldsymbol{\lambda}$,

$$\boldsymbol{\nabla(s_\lambda) = \mathrm{sgn}(\lambda) \sum_{(\Pi,R)\in LNDP_\lambda} t^{\mathrm{area}(\Pi,R)} q^{\mathrm{dinv}(\Pi,R)} x_R,}$$

where

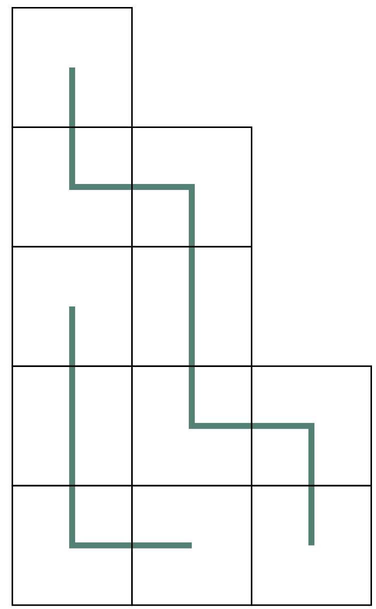
- $\boldsymbol{\Pi = (\pi_0, \ldots, \pi_{\ell(\lambda')-1})}$ is a tuple of **N**ested **D**yck **P**aths
- $\boldsymbol{R = (r_0, \ldots, r_{\ell(\lambda')-1})}$ is a tuple of **L**abels.

Conjectural Expansions of $\nabla(s_\lambda)$

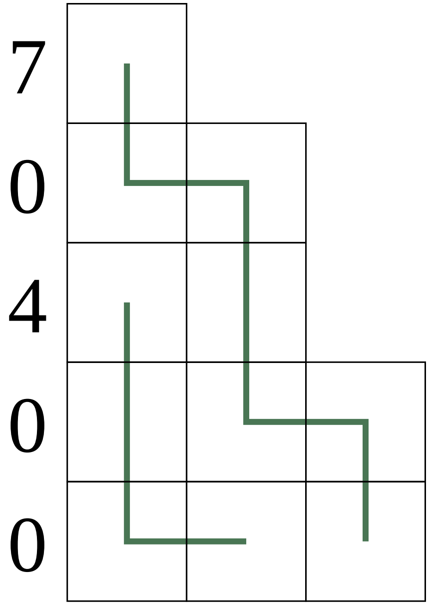
$$\nabla(s_\lambda) = \sum_{\mu} \boxed{\text{labeled, nested Dyck paths}}\ m_\mu$$

$$\nabla(s_\lambda) = \boxed{\text{nested Dyck paths}}\ s_{1^n} + \sum_{\mu \neq 1^n} \boxed{?}\ s_\mu$$

One term of $\nabla(s_{542})|_{m_{111}}$

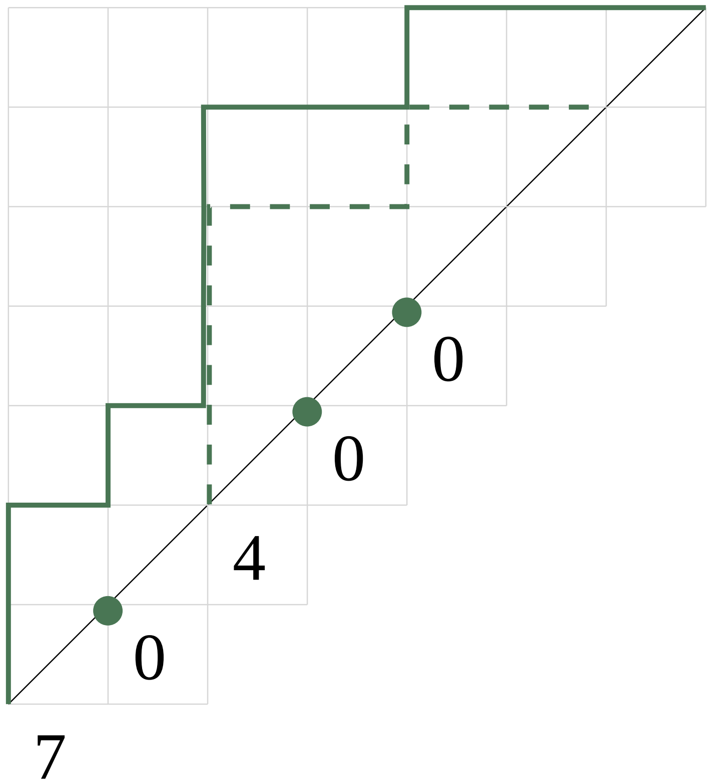


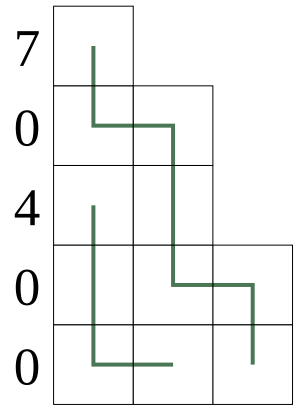
One term of $\nabla(s_{542})|_{m_{111}}$



Terms in $\nabla(s_{542})|_{m_{111}}$ will have positive sign.

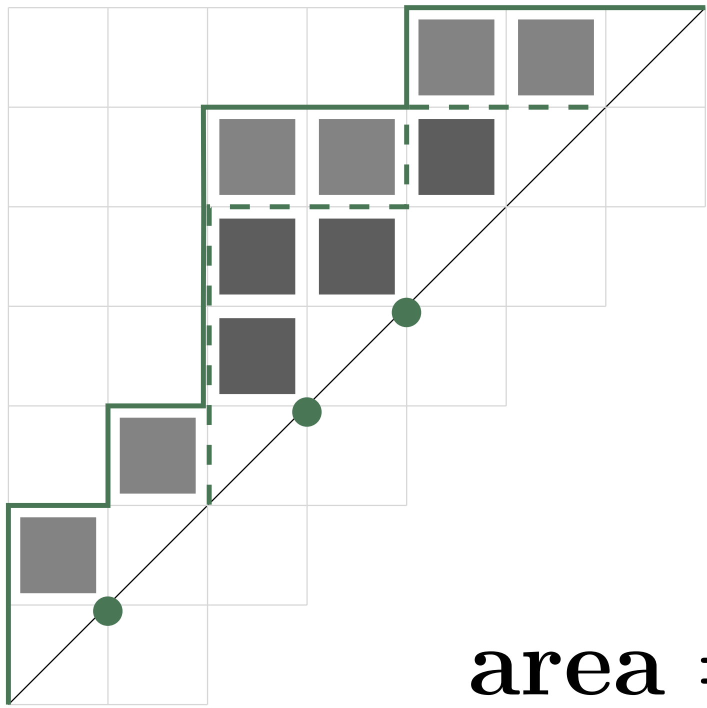
One term of $\nabla(s_{542})\big|_{m_{111}}$





$$(-1)^6 t^{\square} q^{\square} \square$$

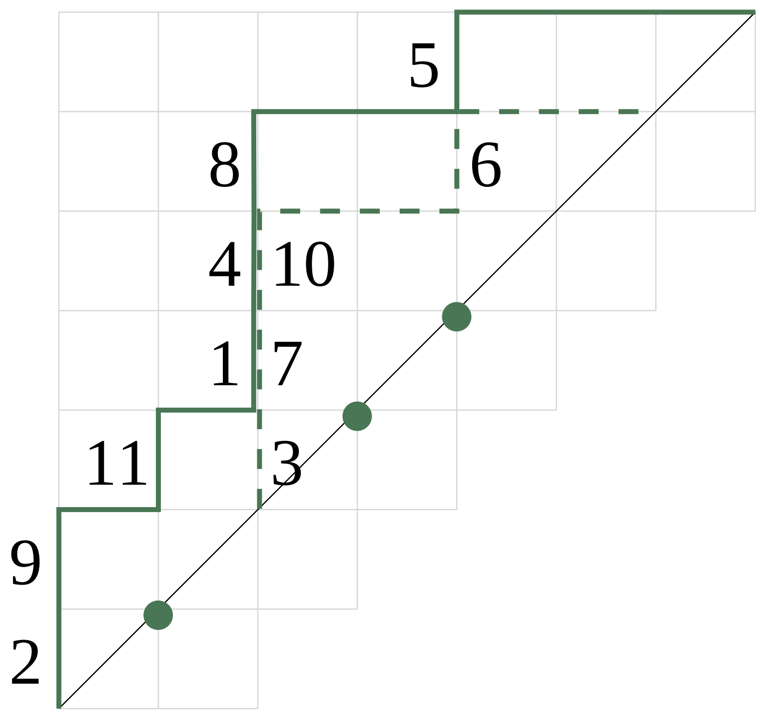
One term of $\nabla(s_{542})|_{m_{111}}$



$$\textbf{area} = 6 \cdot 1 + 4 \cdot 2$$

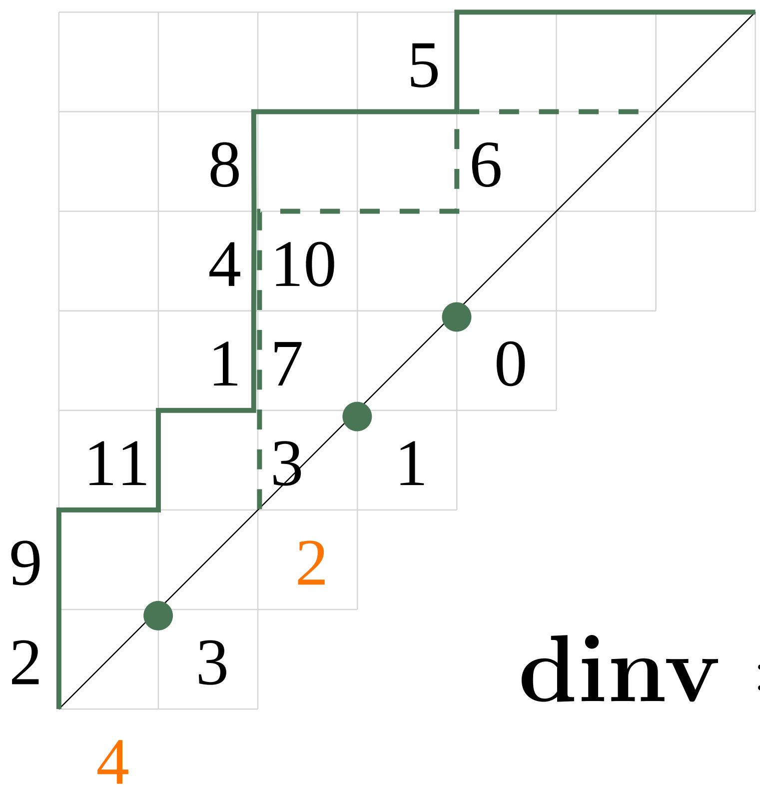
$$(-1)^6 t^{14} q^{\boxed{}} \boxed{}$$

One term of $\nabla(s_{542})|_{m_{111}}$



$$(-1)^6 t^{14} q^{\square} x_1 x_2 \cdots x_{11}$$

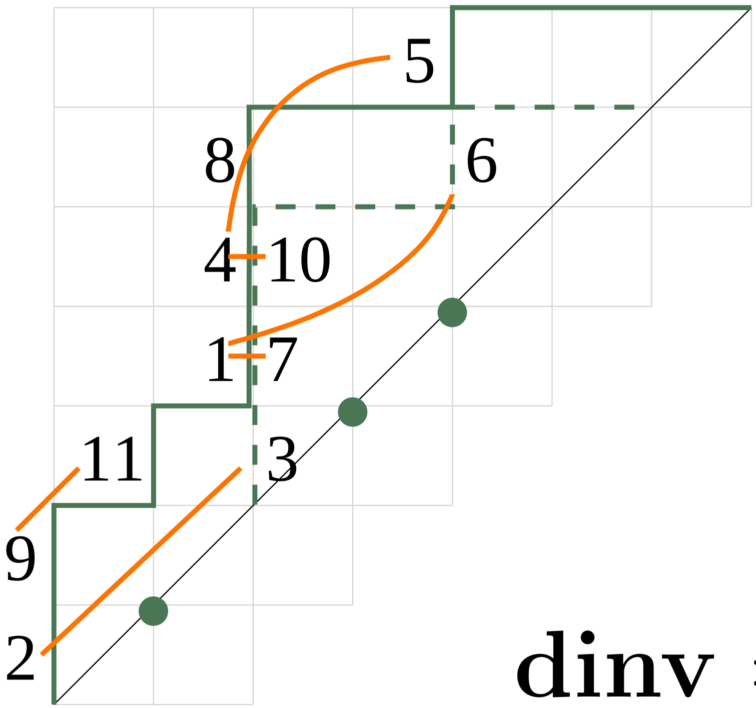
One term of $\nabla(s_{542})|_{m_{111}}$

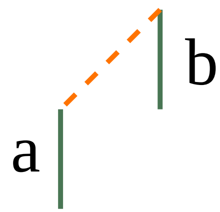


$$\mathbf{dinv} = 6 + $$

$$(-1)^6 t^{14} q^{\square} x_1 x_2 \cdots x_{11}$$

One term of $\nabla(s_{542})|_{m_{111}}$

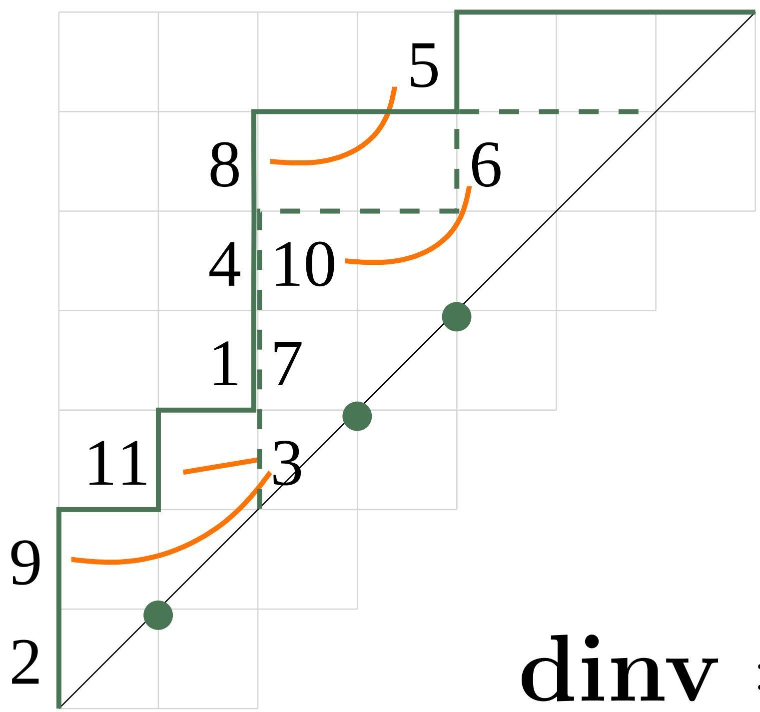


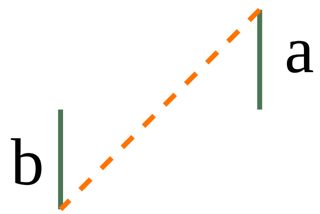


$$\mathbf{dinv} = 6 + 6 +$$

$$(-1)^6 t^{14} q^{\square} x_1 x_2 \cdots x_{11}$$

One term of $\nabla(s_{542})|_{m_{1^{11}}}$





$$\mathbf{dinv} = 6 + 6 + 4$$

$$(-1)^6 t^{14} q^{16} x_1 x_2 \cdots x_{11}$$

$LNDP_\lambda$

$LNDP_\lambda = \{(\Pi, \boldsymbol{R})\}$ as before such that

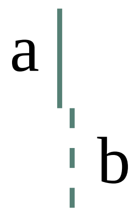
- The i-th path in Π starts at (i, i) and has length equal to that of the i-th hook from the top.

- The entries in i-th label vector in $\boldsymbol{R}$ strictly increase up columns of corresponding path.

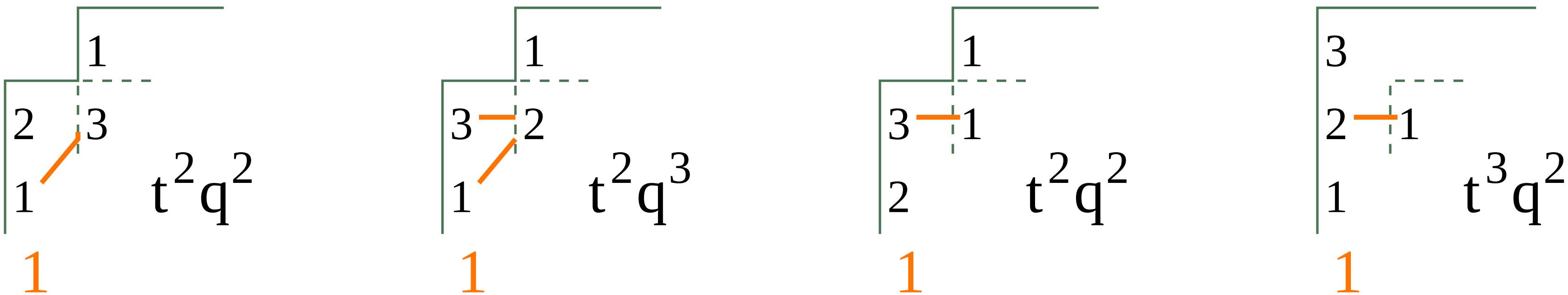
And furthermore. . .

Paths can't

- cross
- share east edges
- pass through another's start
- have $a > b$:



Coefficient of m_{211} in $\nabla(s_{22})$

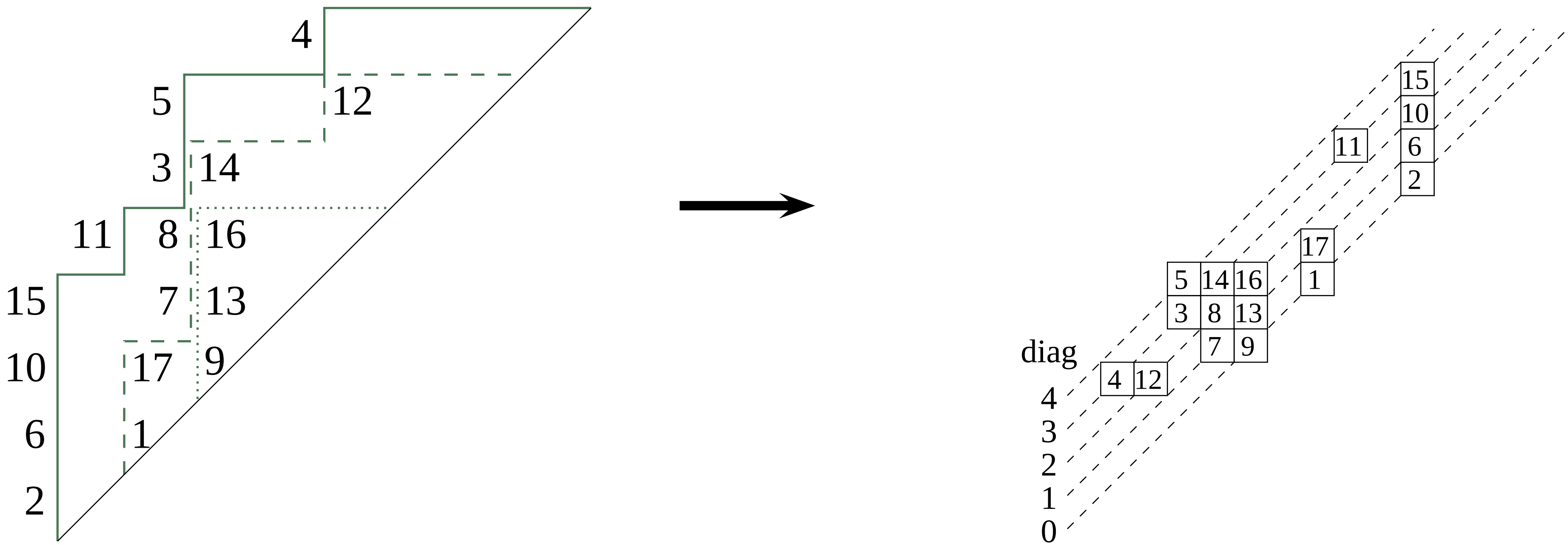


$$\nabla(s_{22}) = -t^2q^2m_{31} - t^2q^2m_{22}$$

$$-t^2q^2(2+t+q)m_{211}$$

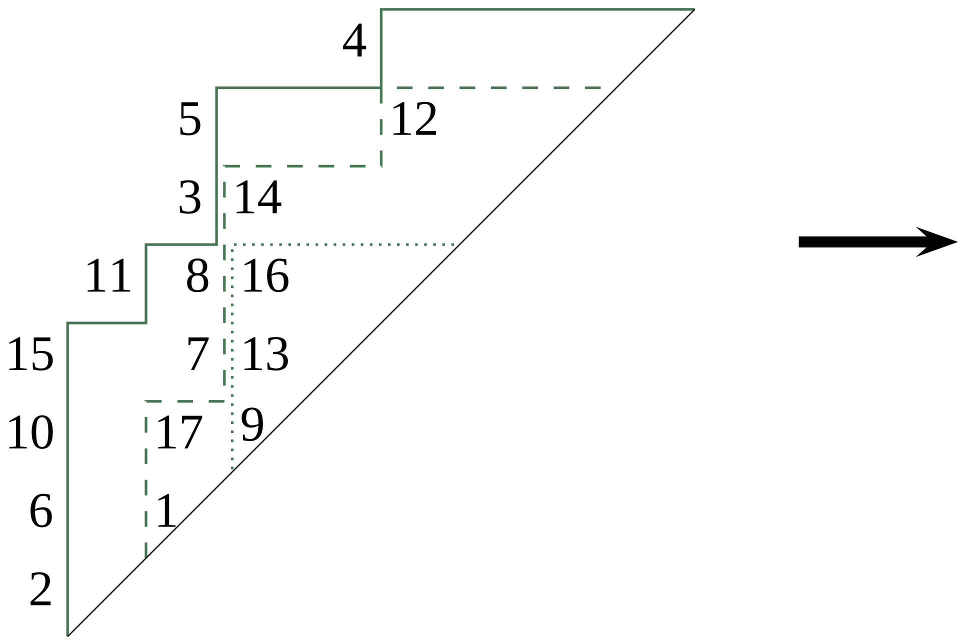
$$-t^2q^2(3t+3q+3+tq)m_{1^4}$$

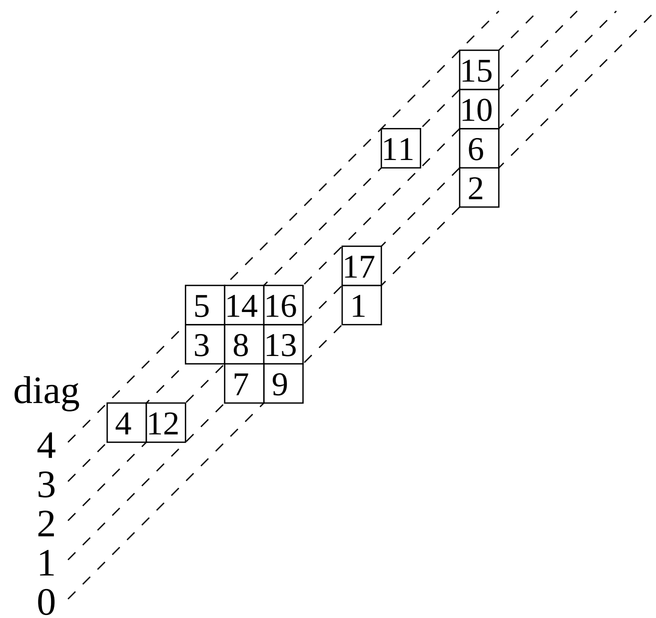
LLT Polynomials



LLT Polynomials

$$\sum_{R:\ (\Pi,R)\in LNDP_\lambda} q^{\mathrm{dinv}(\Pi,R)} x_R = q^{\mathrm{adj}(\lambda)+n(\Gamma(\Pi))} \sum_{T\in SSYT_{\Gamma(\Pi)}} q^{\mathrm{dinv}(T)} x_T$$





Proof when $q = 1$

1. $\boldsymbol{\nabla}_{q=1}$ (i.e., $\tilde{\boldsymbol{\nabla}}$) is a ring homomorphism.

2. (G-H)

$$\boldsymbol{\nabla}_{q=1}(e_n) = \sum_{\pi \in DP_n} t^{\mathrm{area}(\pi)} e_{\alpha(\pi)},$$

where $\alpha(\pi) = (\alpha_1(\pi), \alpha_2(\pi), \ldots)$ gives lengths of columns formed by consecutive north steps.

Proof when $q = 1$

3. By linearity, $\nabla_{q=1}(e_\mu)$ equals

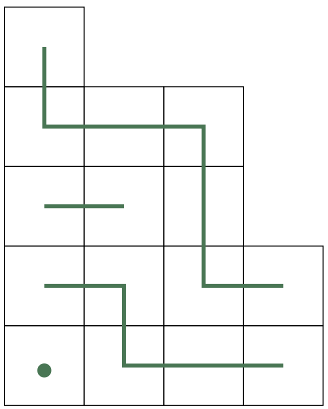
$$\sum_{(\pi_1,\pi_2,\ldots)\in DP_\mu} t^{\sum(\mathrm{area}(\pi_i))} \prod_i e_{\alpha(\pi_i)}.$$

4. $s_\lambda = \sum_\mu K^{-1}_{\mu,\lambda'} e_\mu$.

(R-E) Rim hook tabloids

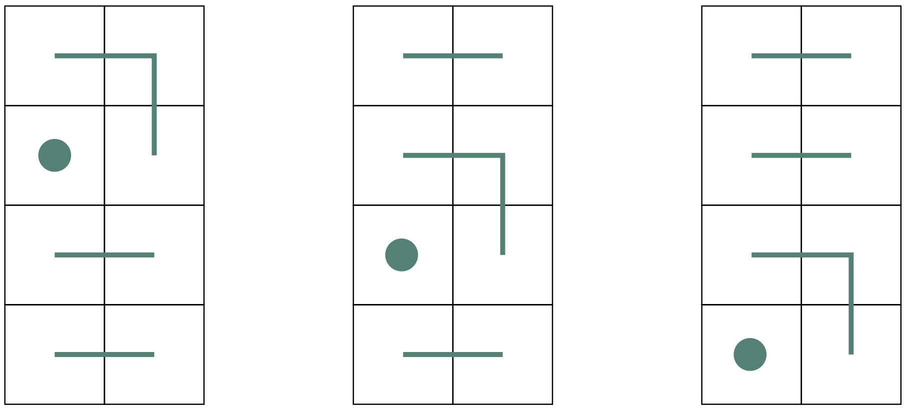
$$K^{-1}_{\mu,\lambda'} = \sum_{T} \mathbf{sgn}(T)$$

where the sum is over all *special rim hook tabloids* T of shape λ' and content μ.



Inverse Kostka matrix

$$s_\lambda = \sum_\mu K^{-1}_{\mu,\lambda'} e_\mu$$

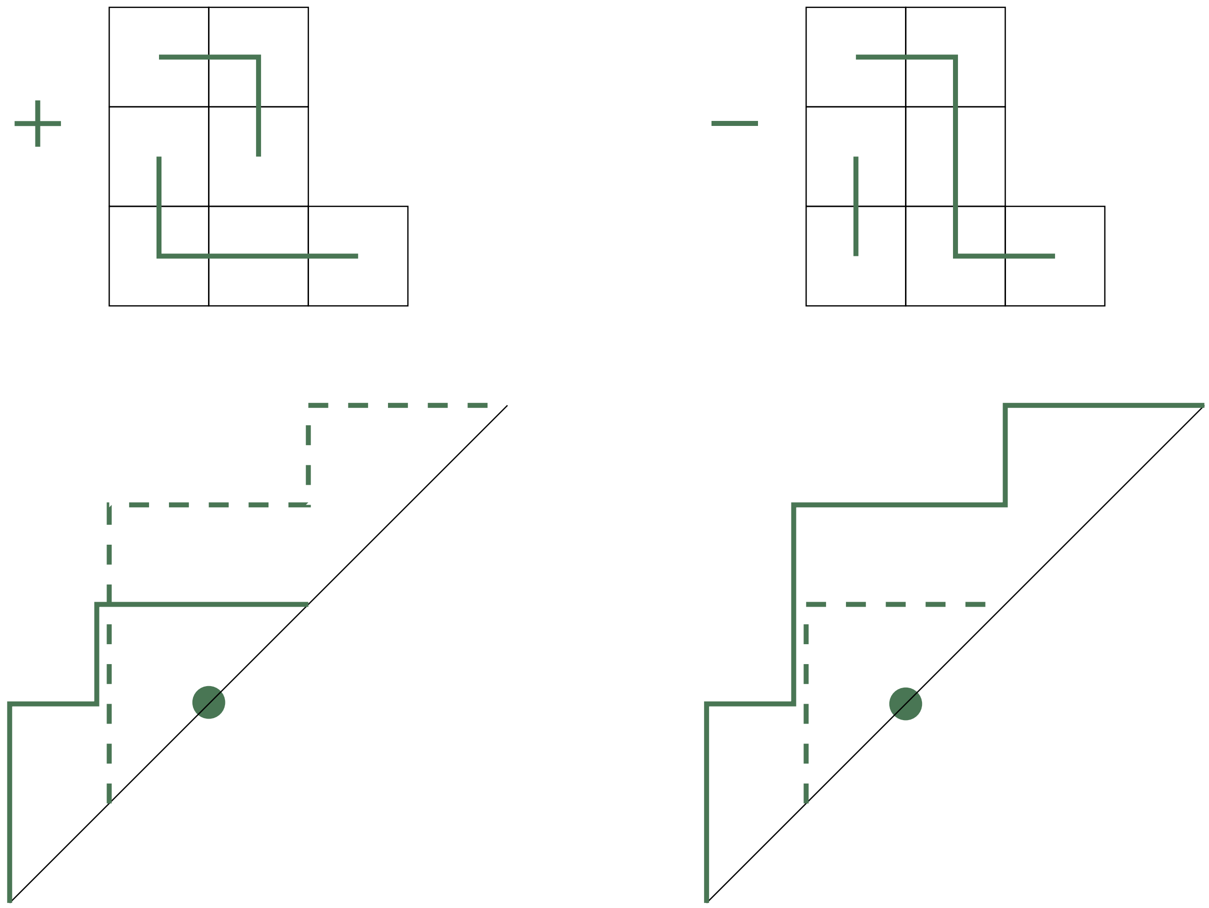


$$s_{44} = \cdots + -3e_{3221} + \cdots$$

Proof when $q = 1$

$$
\begin{aligned}
\nabla_{q=1}(s_\lambda) &= \sum_{\mu} K^{-1}_{\mu,\lambda'} \nabla_{q=1}(e_\mu) \\
&= \sum_{\mu} K^{-1}_{\mu,\lambda'} \sum_{(\pi_i)\in DP_\mu} t^{\sum \text{area}(\pi_i)} \prod_i e_{\alpha(\pi_i)} \\
&= \sum_{\mu} \sum_{T} \text{sgn}(T) \sum_{(\pi_i)\in LDP_\mu} t^{\sum \text{area}(\pi_i)}.
\end{aligned}
$$

Cancellation



$LNDP_\lambda$ conditions

- If $g_{i+1}^{(j)} = g_i^{(j)} + 1$, then $r_i^{(j)} < r_{i+1}^{(j)}$.
- The value $g_i^{(j)}$ is undefined or greater than zero for all $j < i \leq l - 1$.
- For all a and all $j < k$, either one of $g_a^{(j)}$ or $g_{a-1}^{(k)}$ is undefined, or $g_a^{(j)} > g_{a-1}^{(k)}$.
- If $g_a^{(j)}$ and $g_{a-1}^{(k)}$ are defined with $g_a^{(j)} = g_{a-1}^{(k)} + 1$, then $r_a^{(j)} \leq r_{a-1}^{(k)}$.

Dinv statistic

$$\mathrm{dinv}(G,R) = \mathrm{adj}(\lambda) + \sum_{\substack{u,v \\ a \le b}} \chi(g_a^{(u)} - g_b^{(v)} = 1)\chi(r_a^{(u)} > r_b^{(v)})\chi(a \le b) + \sum_{\substack{u,v,a,b \\ a<b \text{ or } (a=b \text{ and } u<v)}} \chi(g_a^{(u)} - g_b^{(v)} = 0)\chi(r_a^{(u)} < r_b^{(v)}).$$