Kazhdan-Lusztig Polynomials for 321-hexagon-avoiding Permutations

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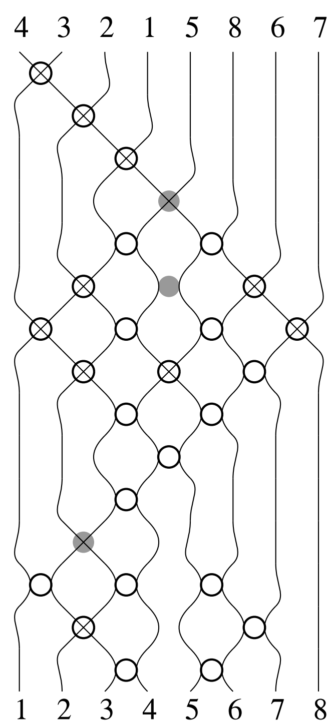
Special Session in Memory of Gian-Carlo Rota



Why...

are K-L polynomials combinatorially interesting?

1. Simple recursive definition
2. Many open problems and conjectures
3. Non-negative coefficients

are K-L polynomials important?

Connect geometry of flag manifold with representations of semisimple Lie groups.

1. **Geometry:** The $P_{x,w}(q)$ are the Poincaré polynomials of local intersection homology groups associated to the Schubert variety X_w.

2. **Representation theory:** The $P_{x,w}(1)$ give multiplicities in Jordan-Hölder series of Verma modules.

Main result

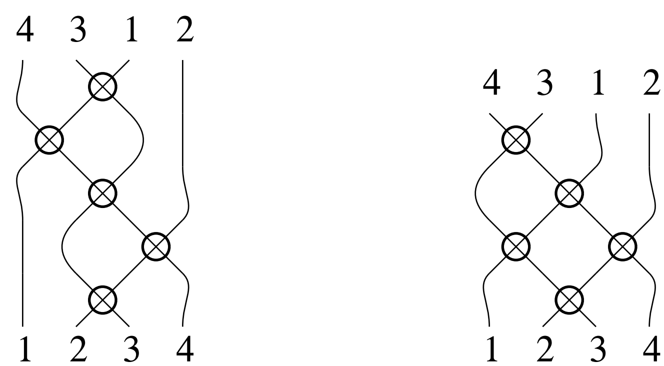
We give an explicit combinatorial formula for the Kazhdan-Lusztig polynomials $P_{x,w}(q)$ when w is a 321-hexagon-avoiding permutation and x is arbitrary.

Symmetric group

$$S_n = \langle s_i : 1 \le i < n,\ \ s_i^2 = 1, \quad (s_i s_{i+1})^3 = 1, \\ (s_i s_j)^2 = 1 \text{ if } |i-j| > 1 \rangle.$$

S_n can be viewed as permutations on n letters via $s_i \mapsto (i, i+1)$. Given some $w = s_{i_1} \cdots s_{i_r}$, we can construct **string diagrams** such as the following:

Example 1. $w = s_2 s_3 s_2 s_1 s_2$ $\qquad$ $w = s_2 s_3 s_1 s_2 s_1$.



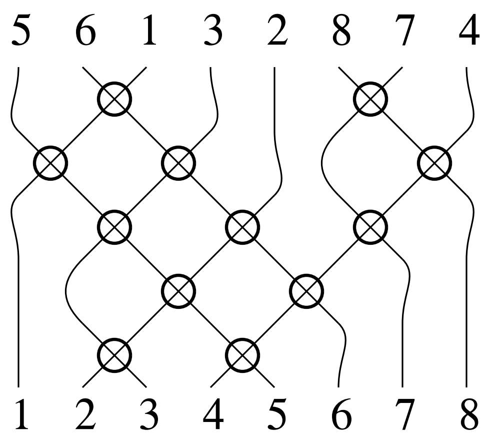
The **length** of w (denoted by $l(w)$) is the:

- minimum # of crossings in a string diagram for w.
- minimum r such that $w = s_{i_1} \cdots s_{i_r}$.

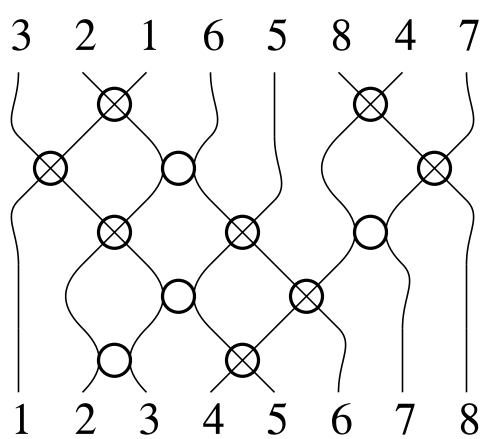
Bruhat-Chevalley order

Definition 2. *$v \leq w$ in the **Bruhat-Chevalley order** iff a string diagram for v can be obtained by erasing some crossings in a string diagram for w.*

w:



v:

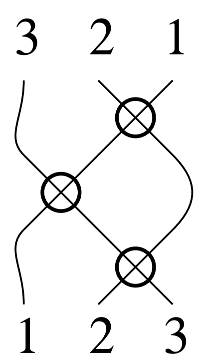


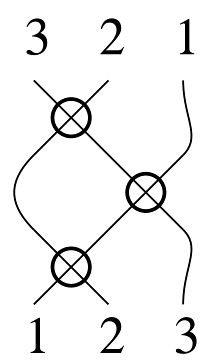
321-avoiding permutations

Definition 3. *$w = [w(1), \ldots, w(n)]$ is **321-avoiding** if $\nexists\ 1 \leq i < j < k \leq n$ such that $w(i) > w(j) > w(k)$.*

- $w = [4, 5, 6, 1, 2, 3]$ is 321-avoiding.
- $w = [3, 5, 1, 4, 2]$ is not 321-avoiding because of the [5,4,2] sequence.

Theorem 4. [B-J-S, '93] *If w is 321-avoiding, then every string diagram for w avoids the following shapes:*



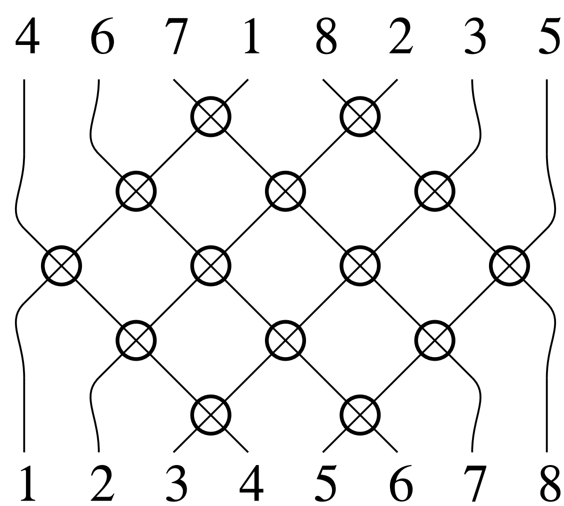


321-hexagon-avoiding permutations

Definition 5. *w is **321-hexagon-avoiding** if it avoids the following patterns:*

$$\begin{array}{ll} [3,2,1] & \\ [5,6,7,8,1,2,3,4] & [4,6,7,8,1,2,3,5] \\ [5,6,7,1,8,2,3,4] & [4,6,7,1,8,2,3,5] \end{array}$$

Fact 6. *If w is 321-avoiding, then w is 321-hexagon-avoiding iff the string diagram for w avoids the following shape:*



Recursive definition of $P_{x,w}(q)$

Let $ws < w \in S_n$ where $s = s_i$ for some i.
The Kazhdan-Lusztig polynomial associated to $x, w \in S_n$ is defined to be

$$P_{x,w}(q) = q^{1-c} P_{xs,ws}(q) + q^c P_{x,ws}(q) - \sum_{\substack{z \\ zs<z}} \mu(z, ws) q^{\frac{l(w)-l(z)}{2}} P_{x,z}(q),$$

where

$$c = \begin{cases} 1, & \text{if } xs < x, \\ 0, & \text{if } xs > x, \end{cases}$$

and $\mu(z, ws)$ denotes the coefficient of

$$q^{(l(ws)-l(z)-1)/2} \text{ in } P_{z,ws}(q).$$

- $P_{w,w}(q) = 1$.
- $P_{x,w}(q) = 0$ if $x \not\leq w$.

Deodhar's framework

For the remainder of the talk, fix some 321-hexagon-avoiding w along with an expression $w = s_{i_1} \cdots s_{i_{l(w)}}$.

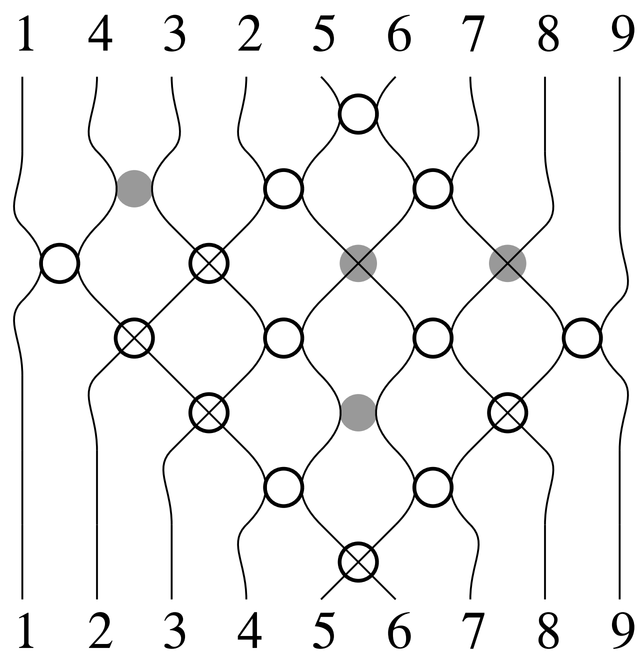
Definition 7. *Let $\mathcal{P}(w)$ denote the set of all subexpressions of $s_{i_1} \cdots s_{i_{l(w)}}$. An element $\boldsymbol{\sigma} \in \mathcal{P}(w)$ is called a* $\mathbf{mask}$*.*

Example 8. $w = s_2 s_1 s_4 s_3 s_2$ *and* $\boldsymbol{\sigma} = (1, 0, 1, 0, 0)$*.*

Definition 9. $\pi(\boldsymbol{\sigma})$ *denotes the permutation corresponding to $\boldsymbol{\sigma}$ obtained by reading top row of string diagram.*

Defect set

Definition 10. *The* **defect set** $\mathcal{D}(\boldsymbol{\sigma})$ *is the set of vertices at which two strings meet that have previously crossed an odd number of times.*



Theorem 11. [D, '90] *Let* $w \in S_n$. *Then* $\exists\ \mathcal{S}(w) \subseteq \mathcal{P}(w)$ *such that* $\forall\ x \in W$,

$$P_{x,w}(q) = \sum_{\substack{\boldsymbol{\sigma} \in \mathcal{S}(w) \\ \pi(\boldsymbol{\sigma}) = x}} q^{|\mathcal{D}(\boldsymbol{\sigma})|}.$$

Main result

Theorem 12. [B-W, '99] *Let $w \in S_n$. TFAE:*

1. *w is 321-hexagon-avoiding.*

2. $P_{x,w}(q) = \sum_{\substack{\boldsymbol{\sigma} \in \mathcal{P}(w) \\ \pi(\boldsymbol{\sigma}) = x}} q^{|\mathcal{D}(\boldsymbol{\sigma})|}$,

3. *The Poincaré polynomial for the full intersection cohomology group of X_w is*

$$\sum_i \dim(\mathrm{IH}^{2i}(X_w))q^i = \sum_{x \leq w} q^{l(x)} P_{x,w}(q) = (1+q)^{l(w)}.$$

4. *The Kazhdan-Lusztig basis element C'_w satisfies $C'_w = C'_{s_{i_1}} \cdots C'_{s_{i_{l(w)}}}$.*

5. *The Bott-Samelson resolution of X_w is small.*

6. $\mathrm{IH}_*(X_w) \cong H_*(Y)$, *where Y is the Bott-Samelson resolution of X_w.*

Highlights/Corollaries

In the case where w is 321-hexagon-avoiding, we get:

- A combinatorial object which $P_{x,w}(q)$ counts.
- Geometric information phrased in terms of pattern avoidance.
- An explicit description of the maximal singular locus of X_w.
- A nice formula for the Poincaré polynomial of $IH(X_w)$.
- The 0–1 conjecture posits that $\mu(x, w) \leq 1$ for all $x, w \in S_n$. This conjecture is true when w is 321-hexagon-avoiding.
- The equivalence of a number of notions.

Generalizations

- The theory applies when $W = B_n, C_n, F_4$ or G_2.
- Can be extended to $\widehat{A}_n$.
- There appear to be 321-hexagon-avoiding analogues in E_6 and D_n that can also be expressed succinctly in terms of pattern avoidance.
- The defect method is promising for arbitrary permutations w.

Enumeration

The following table lists both the number of 321-avoiding elements in S_n ($\frac{1}{n+1}\binom{2n}{n}$) and the number of 321-hexagon-avoiding elements in S_n for $7 \leq n \leq 13$ (these numbers are equal for $n \leq 6$).

	321-avoiding	321-hexagon-avoiding
7	429	429
8	1430	1426
9	4862	4806
10	16796	16329
11	58786	55740
12	208012	190787
13	742900	654044

Related results

- Lascoux & Schützenberger — combinatorial formula for Grassmannian permutations
- Boe — extension of L-S result to other Weyl groups
- Dyer — formula when W is a universal Coxeter group
- Brenti — general formula in terms of Bruhat graph
- Brenti-Simion — formulas in specific cases
- Polo — every polynomial $p(q) \in \mathbb{N}[q]$ with $p(0) = 1$ can be realized as a $P_{x,w}(q)$ for some $x, w \in S_n$ and some n.