The 0,1-Conjecture

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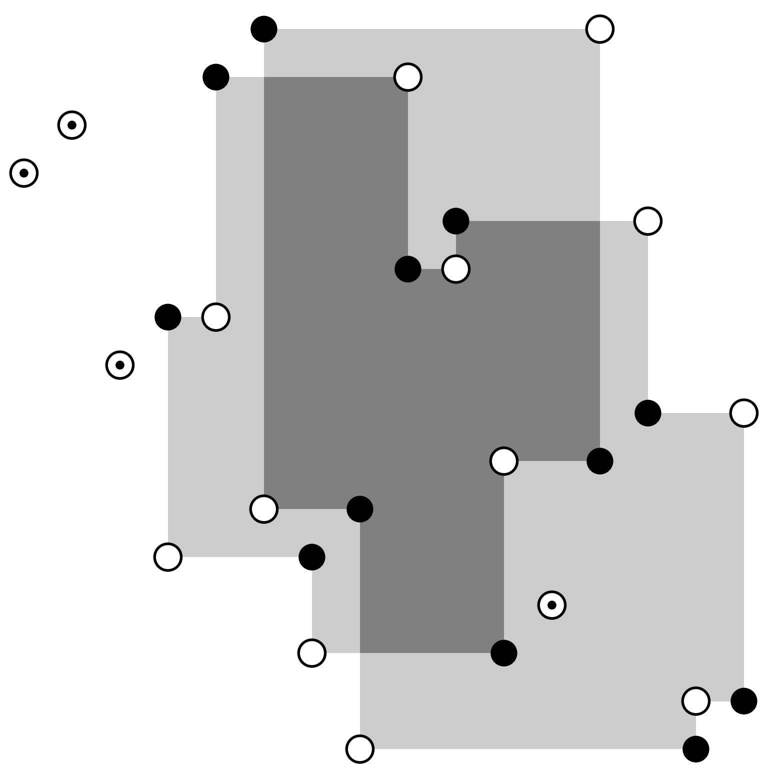


Outline

1. The conjecture
2. Kazhdan-Lusztig polynomials
3. Representations of Hecke algebras
4. McLarnan's counterexamples
5. Smaller counterexamples
6. 321,hexagon-avoiding permutations

The Conjecture

Conjecture 1. For any $x, w \in S_n$, the coefficient of the term of maximum possible degree in the corresponding Kazhdan-Lusztig polynomial $P_{x,w}$ is 0 or 1.

The analogous conjecture fails in B_5, D_6, E_6 and H_4:

1. In B_5, $P_{1212454,2123432123543212} =$
   $2q^4 + 6q^3 + 7q^2 + 5q + 1.$

2. In D_6, $P_{123123565,3123431235465431 23} =$
   $2q^4 + 10q^3 + 15q^2 + 8q + 1.$

3. In E_6, $P_{1231565,12345654231435} =$
   $3q^3 + 7q^2 + 5q + 1.$

4. In H_4, $P_{121343,12343212134} =$
   $2q^2 + 3q + 1.$

Other types

1. In B_5, $P_{1212454,2123432123543212} =$
   $2q^4 + 6q^3 + 7q^2 + 5q + 1.$

2. In D_6, $P_{123123565,312343123546543123} =$
   $2q^4 + 10q^3 + 15q^2 + 8q + 1.$

3. In E_6, $P_{1231565,12345654231435} =$
   $3q^3 + 7q^2 + 5q + 1.$

4. In H_4, $P_{121343,12343212134} =$
   $2q^2 + 3q + 1.$

K-L polynomials

For any Coxeter group W, Kazhdan and Lusztig defined polynomials $P_{x,w}$ for each pair $x, w \in W$.

Can be approached from:

1. Representation theory
   - Representations of Hecke algebras
   - Multiplicities in Verma modules
2. Geometry
   - Singularities of Schubert varities
3. Combinatorics
   - Intrinsic proofs of properties

Hecke algebras

Let (W, S) be a Coxeter system.

The $\mathcal{H}$ Hecke algebra associated to W is the $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-algebra with basis $\{T_w\}_{w \in W}$ and multiplication:

$$T_s T_w = \begin{cases} T_{sw}, & sw > w, \\ (q-1)T_w + qT_{sw}, & sw < w. \end{cases}$$

There is an involution $^{-}$ on $\mathcal{H}$ which is the linear extension of $q \mapsto q^{-1}$ and $T_w \mapsto (T_{w^{-1}})^{-1}$.

Theorem 2 (K-L). There is a unique basis $\{C_w\}_{w \in W}$ of $\mathcal{H}$ such that

1. $\overline{C_w} = C_w$, and

2. $C_w = q^{\frac{l(w)}{2}} \sum_{x \le w} (-1)^{l(w)-l(y)} q^{-l(y)} \overline{P_{x,w}} T_x$,

where $P_{w,w} = 1$ and $2\deg(P_{x,w}) \le (l(w) - l(x) - 1)$ when $x < w$.

Properties of $P_{x,w}$

1. $P_{x,w} = 0$ if $x \not\leq w$.

2. $P_{w,w} = 1 \ \forall w \in W$.

3. $P_{x,w}(0) = 1$ if $x \leq w$.

4. $P_{sx,w} = P_{x,w}$ if $sw < w$.

Let $[q^k]P_{x,w}$ denote the coefficient of q^k in $P_{x,w}$ and let $\mu(x, w) = [q^{(l(w)-l(x)-1)/2}]P_{x,w}$. Let w_0 denote the element of maximal length in W (if there is one).

1. $\mu(w_0 w, w_0 x) = \mu(x, w) = \mu(w w_0, x w_0)$.

2. $sw < w$ and $sx > x$ implies $\mu(x, w) = 0$.

3. For $sw < w$, $P_{x,w} = qP_{x,sw} + P_{sx,sw} -$

$$\sum_{\substack{z \leq sw \\ sz < z}} \mu(z, sw) q^{\frac{l(w)-l(z)}{2}} P_{x,z} \text{ if } sx < x.$$

A Representation of $\mathcal{H}$

Theorem 3.

$$T_s C_w = \begin{cases} -C_w, & sw < w, \\ q^{1/2} C_{sw} + q C_w + q^{1/2} \sum_{sz<z} C_z, & sw > w. \end{cases}$$

Definition 4. For x, w comparable, set

$$\mu[x, w] = \begin{cases} \mu(x, w), & x \leq w, \\ \mu(w, x), & w < x. \end{cases}$$

Definition 5. Let $M(s)$ be the transition matrix:

$$M(s)_{ww} = \begin{cases} -1, & sw < w. \\ q, & sw > w. \end{cases}$$

$$M(s)_{xw} = \begin{cases} q^{1/2} \mu[x, w], & sx < x. \\ 0, & \text{otherwise.} \end{cases}$$

Set $M(w) = M(s_{i_1}) M(s_{i_2}) \cdots M(s_{i_r})$ when w has a reduced expression $s_{i_1} s_{i_2} \cdots s_{i_r}$.

L-Cells

Fact 6. For $x \neq w$, C_x occurs in the expansion of $T_s C_w$ for some $s \in S$ if and only if:

1. $D_L(x) \not\subseteq D_L(w)$ and
2. $\mu(x, w) \neq 0$.

We make the following definitions:

1. $w \xrightarrow{L} x$ if 1 and 2 hold.
2. An **L-path** is a sequence

$$w \xrightarrow{L} x_1 \xrightarrow{L} x_2 \xrightarrow{L} \cdots \xrightarrow{L} x_k.$$

3. Equivalence classes under L-paths are called **L-cells**.

Irreducibles ($W = S_n$)

Fact 7. L-cells are parameterized by Standard Young Tableaux of size n.

Theorem 8. Representations obtained by restricting to L-cells (for $W = S_n$) are irreducible. The labelling by partitions is the same as Young's labelling.

McLarnan's counterexamples

T. McLarnan ('88) found the following counterexamples:

$$x = [6, 5, 2, 1, 10, 9, 4, 3, 14, 13, 8, 7, 12, 11, 16, 15]$$
$$w = [13, 9, 2, 1, 14, 10, 5, 3, 16, 11, 6, 4, 12, 7, 15, 8]$$
$$l(x) = 32, \quad l(w) = 53,$$
$$P_{x,w} = 5q^{10} + 72q^9 + 387q^8 + 1039q^7 + 1610q^6 + 1536q^5 + 931q^4 + 365q^3 + 92q^2 + 14q + 1.$$

$$x = [8, 7, 4, 2, 1, 13, 12, 6, 5, 3, 11, 10, 9, 16, 15, 14]$$
$$w = [13, 11, 7, 2, 1, 16, 12, 8, 4, 3, 14, 9, 5, 15, 10, 6]$$
$$l(x) = 39, \quad l(w) = 60,$$
$$P_{x,w} = 5q^{10} + 56q^9 + 231q^8 + 533q^7 + 776q^6 + 755q^5 + 501q^4 + 226q^3 + 67q^2 + 12q + 1.$$

Bruhat pictures

Definition 9.
Define the **rank function** of w by

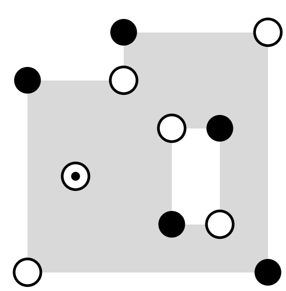
$$r_w(p,q) = |\{i \leq p : w(i) \geq q\}|.$$

Define the **difference function** of x, w by

$$d_{x,w}(p,q) = r_w(p,q) - r_x(p,q).$$

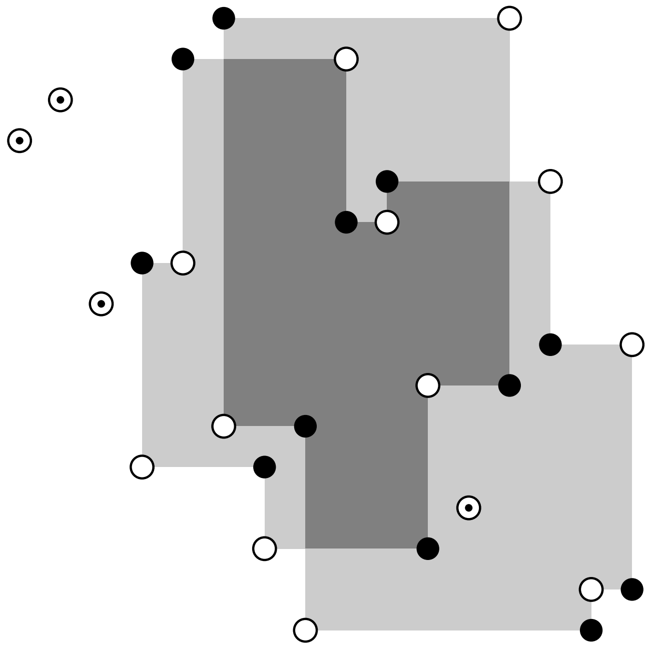
Fact 10. $x \leq w$ if and only if $d_{x,w} \geq 0$.

Definition 11. The **Bruhat picture** associated to x and w is the overlay of the permutation matrices accompanied by shading courtesy of $d_{x,w}$.

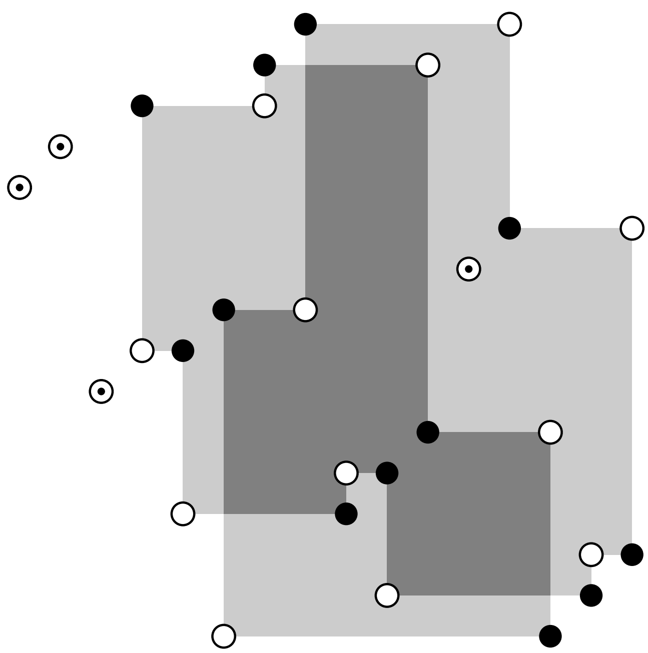


w = [6,3,4,2,5,1]
x = [3,1,5,2,4,6]

T. McLarnan's counterexamples



$1 + 14q + 92q^2 + 365q^3 + 931q^4 + 1536q^5 + 1610q^6 + 1039q^7 + 387q^8 + 72q^9 + 5q^{10}$



$1 + 12q + 67q^2 + 226q^3 + 501q^4 + 755q^5 + 776q^6 + 533q^7 + 231q^8 + 56q^9 + 5q^{10}$

$\widetilde{\ }$ Operator

Definition 12. For $w \in S_n$ and $1 \le i_1 < \cdots < i_k \le n$,

$$v = \mathrm{fl}([w(i_1), \ldots, w(i_k)])$$

is the unique permutation in S_k such that

$$v(j) \le v(k) \Leftrightarrow w(i_j) \le w(i_k).$$

Definition 13. Set

$$\Delta(x, w) = \{i : x(i) \ne w(i) \text{ or } d_{x,w}(i, w(i)) \ne 0\}.$$

For any ordered pair (x, w) with

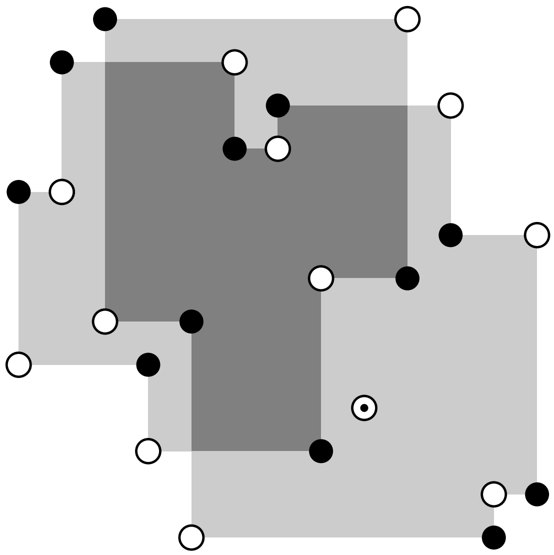
$$\Delta(x, w) = \{i_1 < i_2 < \ldots < i_k\},$$

set

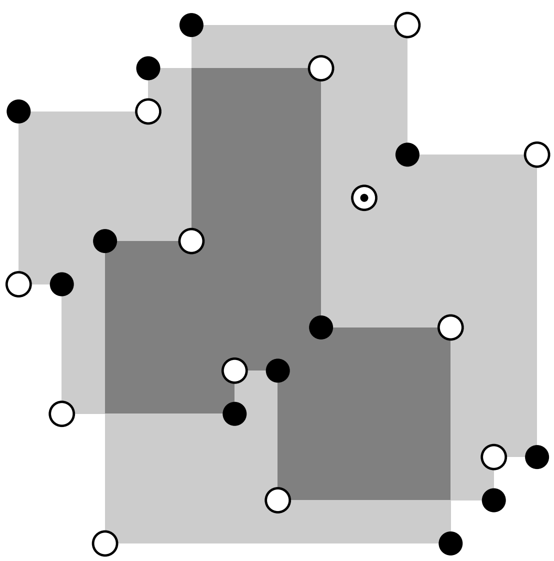
$$\widetilde{x} = \mathrm{fl}([x(i_1), \ldots, x(i_k)]),$$
$$\widetilde{w} = \mathrm{fl}([w(i_1), \ldots, w(i_k)]).$$

Theorem 14. $P_{x,w} = P_{\widetilde{x},\widetilde{w}}$.

Flattened counterexamples



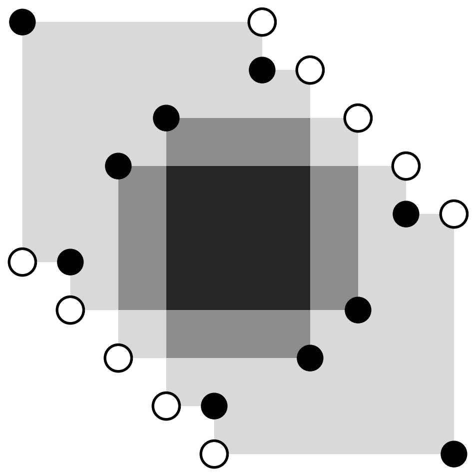
$1 + 14q + 92q^2 + 365q^3 + 931q^4 + 1536q^5 + 1610q^6 + 1039q^7 + 387q^8 + 72q^9 + 5q^{10}$



$1 + 12q + 67q^2 + 226q^3 + 501q^4 + 755q^5 + 776q^6 + 533q^7 + 231q^8 + 56q^9 + 5q^{10}$

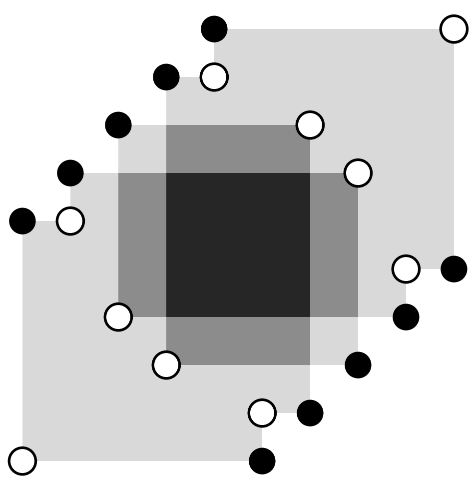
Counterexamples in S_{10}

$w = [6,7,8,9,10,1,2,3,4,5]$
$x = [1,6,4,3,9,2,8,7,5,10]$



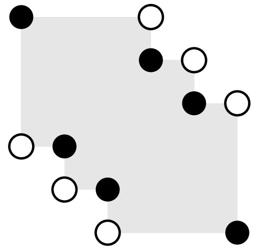
$1 + 16q + 75q^2 + 124q^3 + 58q^4 + 4q^5$

$w = [10,5,7,8,2,9,3,4,6,1]$
$x = [5,4,3,2,1,10,9,8,7,6]$

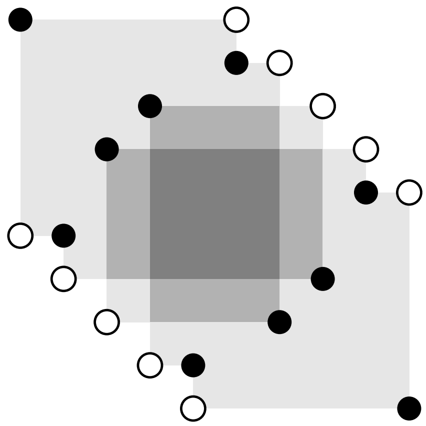


$1 + 7q + 19q^2 + 26q^3 + 17q^4 + 4q^5$

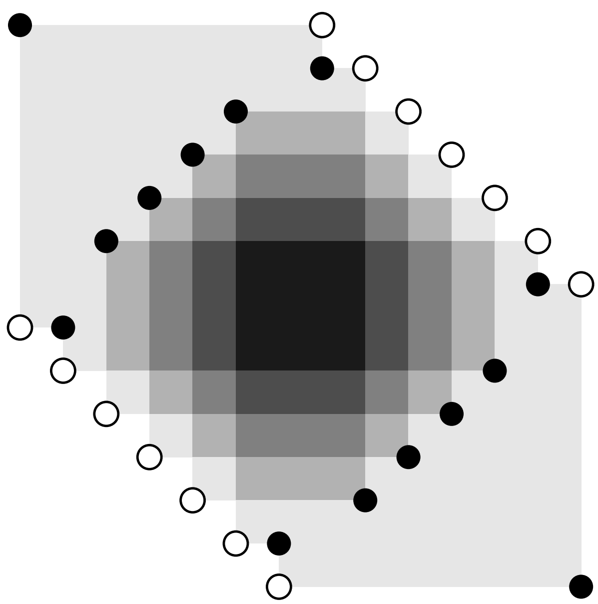
A Family



$1 + 4q + q^2$



$1 + 16q + 75q^2 + 124q^3 + 58q^4 + 4q^5$



$1 + 36q + 455q^2 + 2564q^3 + 6359q^4 + 7090q^5 + 3462q^6 + 620q^7 + 19q^8$

Sequences

$a_n = 4a_{n-1} + 3a_{n-2}$	0,1,4,19,88,409,1900, 8827,41008,190513,...
Powers of $\sqrt{19}$ rounded down	1,4,19,82,361,1573, 6859,29897,130321,...
$\|2^n - n^3\|$	1,1,4,19,48,93,152, 215,256,217,24,717,...
Trees of diameter 8	1,4,19,66,219,645, 1813,4802,12265,...
$n(n+7)$ is a palindrome	0,1,4,19,26,219,664, 902,2109,2972,2982,...
Base 4 digits are first n terms of periodic sequences with initial period 1,0,3	1,4,19,77,308,1235, 4941,19764,79059,...

Definition of L_i and R_j

Suppose $\{w(i), w(i+1), w(i+2)\} = \{\alpha < \beta < \gamma\}$. Let $\mathcal{R}_i$ be the set of permutations w for which $ws_i < w$ xor $ws_{i+1} < w$. Define an operator R_i acting on $\mathcal{R}_i$ by:

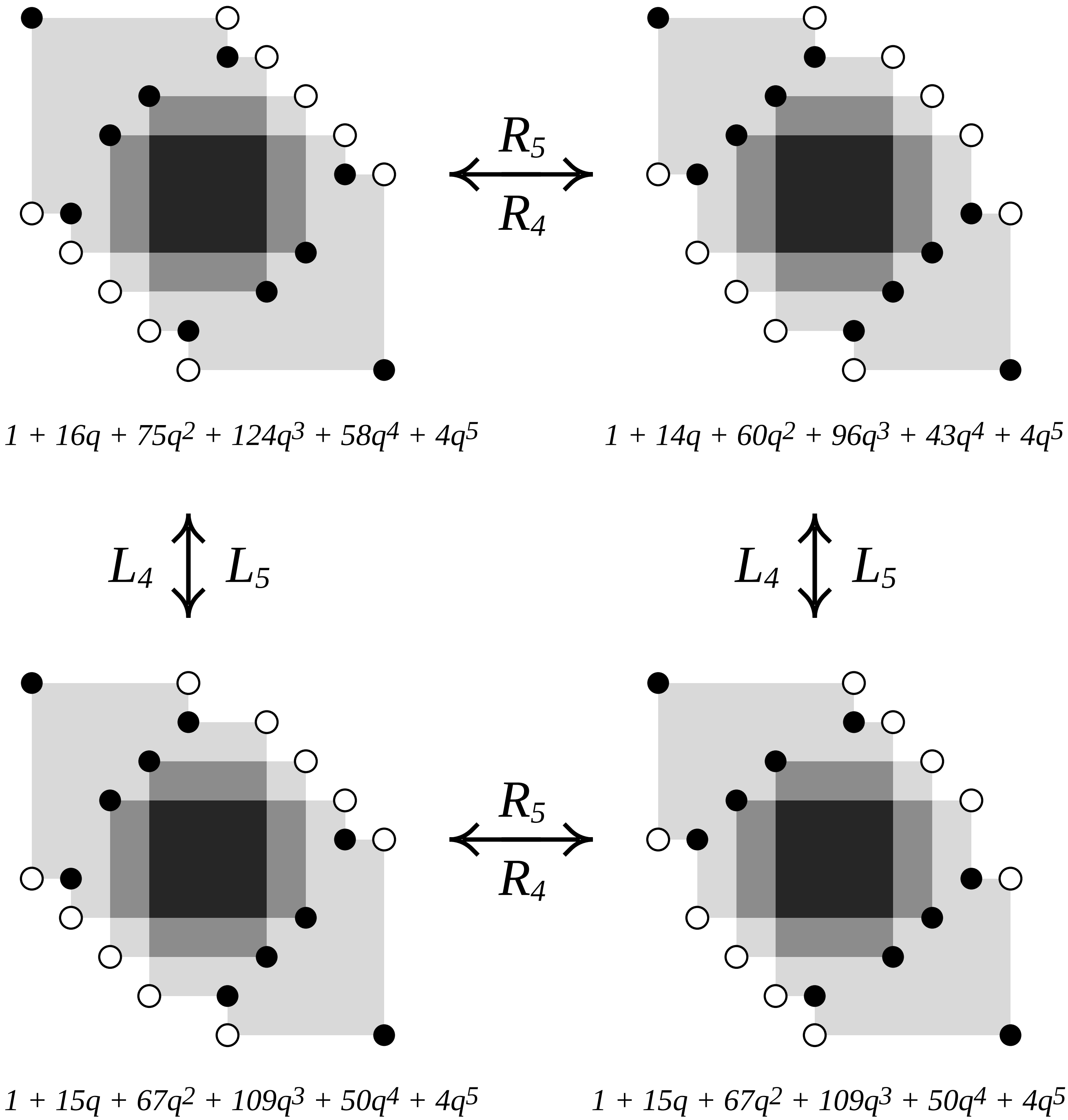
$$\begin{array}{ccccccc} \alpha & \gamma & \beta & \longrightarrow & \gamma & \alpha & \beta \\ \beta & \alpha & \gamma & \longrightarrow & \beta & \gamma & \alpha \\ \beta & \gamma & \alpha & \longrightarrow & \beta & \alpha & \gamma \\ \gamma & \alpha & \beta & \longrightarrow & \alpha & \gamma & \beta \end{array}$$

Define $\mathcal{L}_i$ and L_i (acting on the left) analogously.

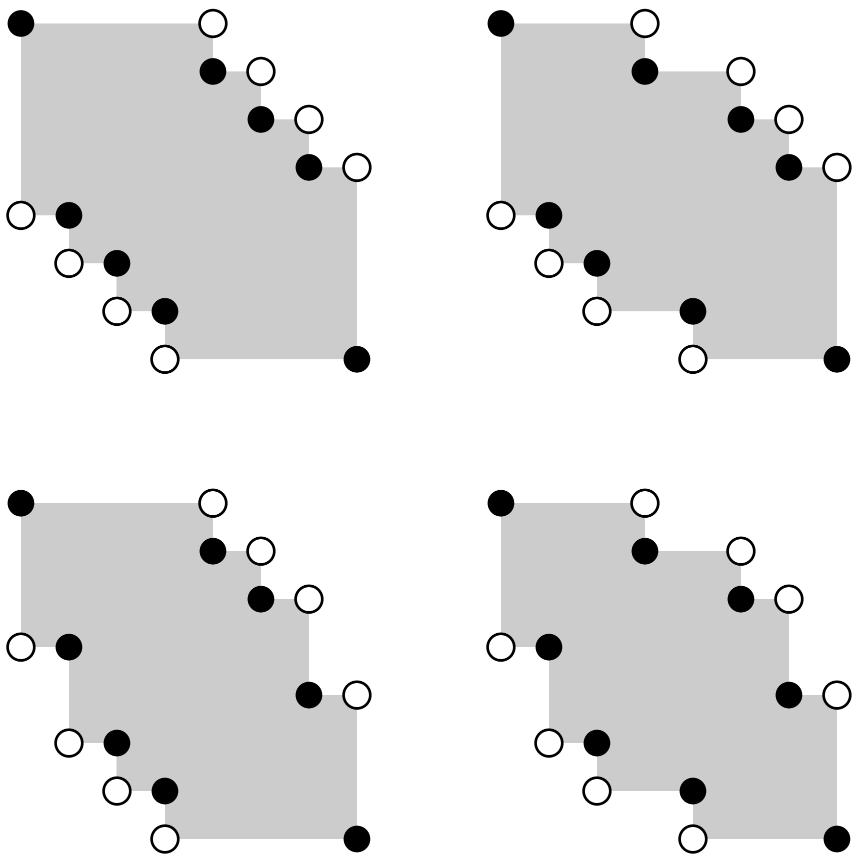
Example 15.
$R_3[4,2,5,1,3] = [4,2,1,5,3]$,
$L_3[4,2,5,1,3] = [4,2,3,1,5]$.

Theorem 16.
If $x, w \in \mathcal{L}_i$, then $\mu(x,w) = \mu(L_i x, L_i w)$.
If $x, w \in \mathcal{R}_i$, then $\mu(x,w) = \mu(xR_i, wR_i)$.

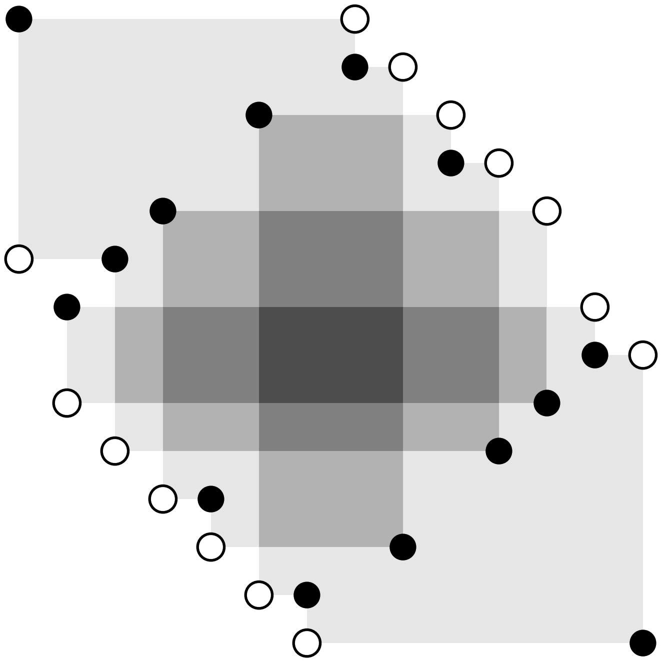
Action of L_i and R_j



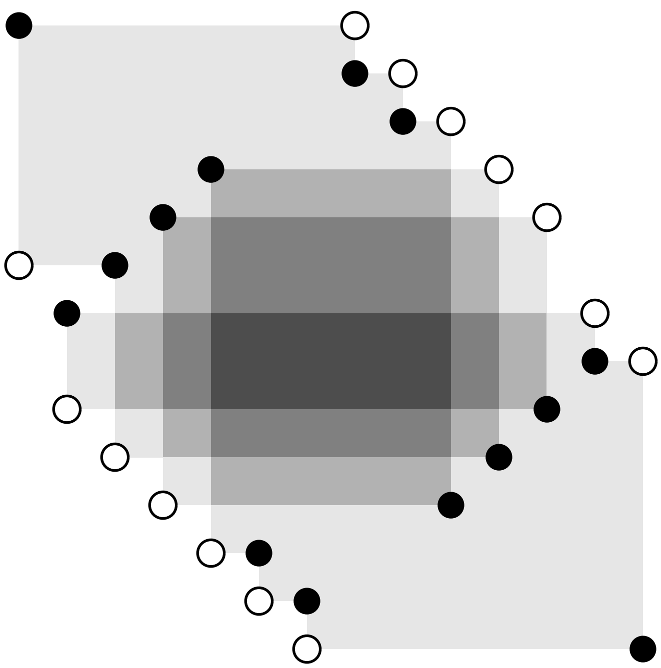
Hexagons



A Two and a Three



$1 + 29q + 257q^2 + 908q^3 + 1292q^4 + 693q^5 + 111q^6 + 2q^7$



$1 + 29q + 263q^2 + 960q^3 + 1346q^4 + 716q^5 + 124q^6 + 3q^7$