Quasisymmetric expansions of cycle indices

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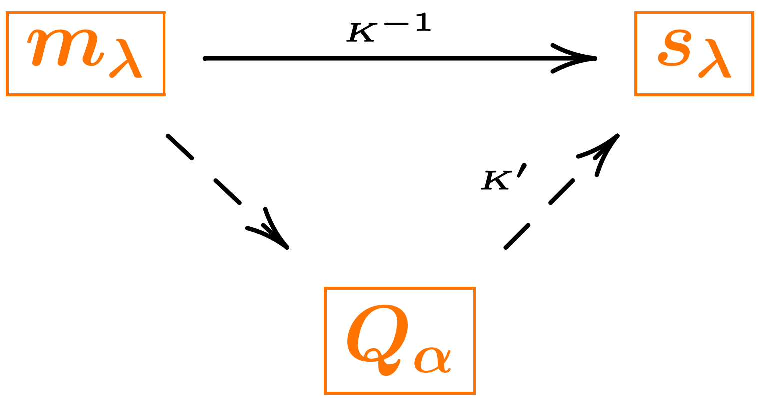

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AMS Special Session

The Combinatorics of Symmetric Functions

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Outline

1. $\{x, M, m, Q\}$-expansions of the pattern inventory

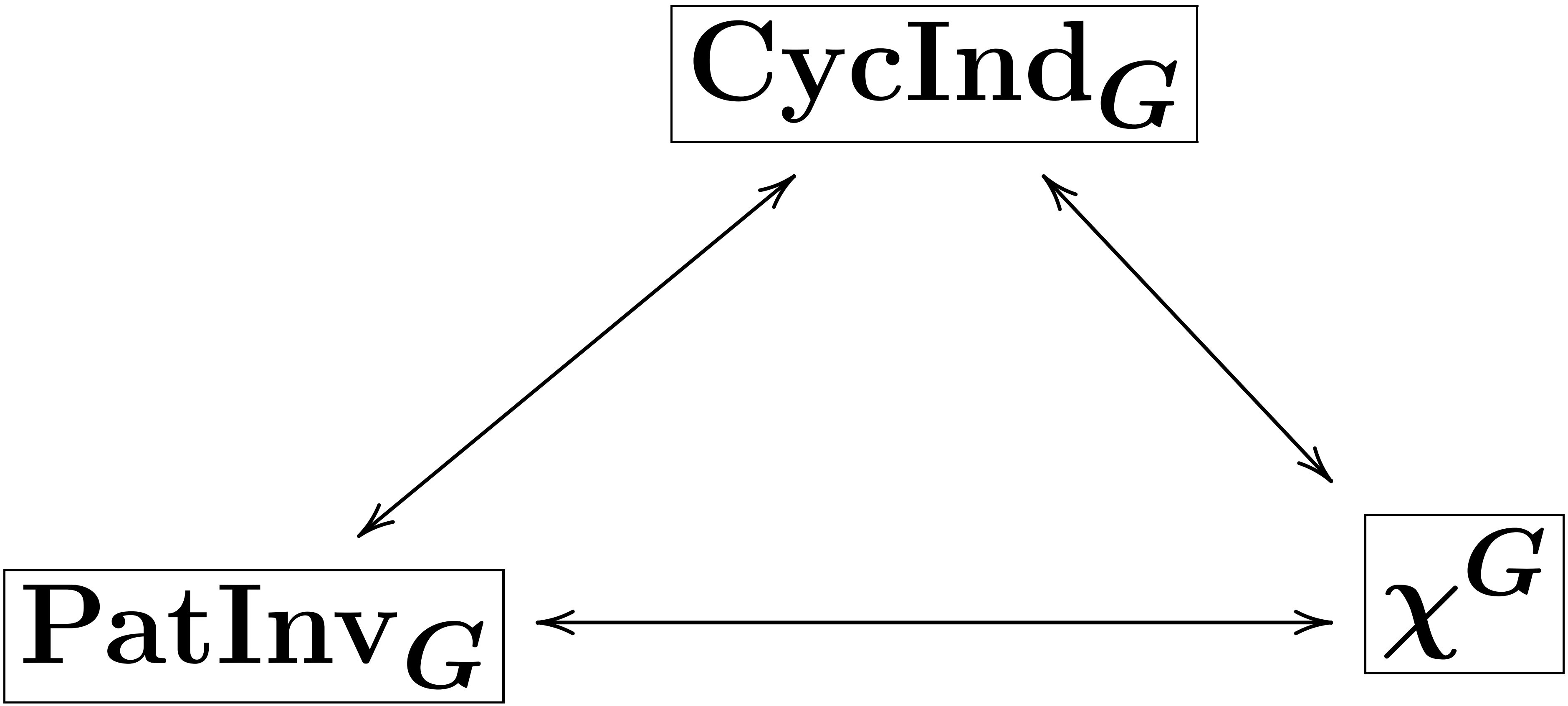
2. Standardization

3. General strategy

4. Examples

5. Representation theory

Pólya Theory



Pattern Inventory

Let $G \leq S_n$ act on a set X.

Let C be a set of colors.

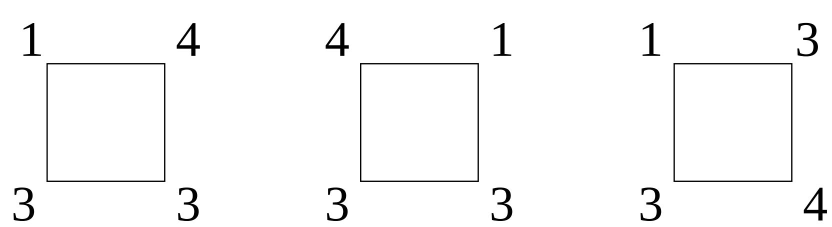
Define the pattern inventory

$$\mathrm{PatInv}_G(x) = \sum_{\mathcal{O} \in \{f : X \to C\}/G} x_{\mathrm{wt}(\mathcal{O})}.$$

x-expansion of PatInv_G

- Pick a set $X =$ {corners of a square},
- colors $C = \{1, 2, 3, 4\}$, and a
- group $G = C_4$ (rotations by $k\pi/2$).

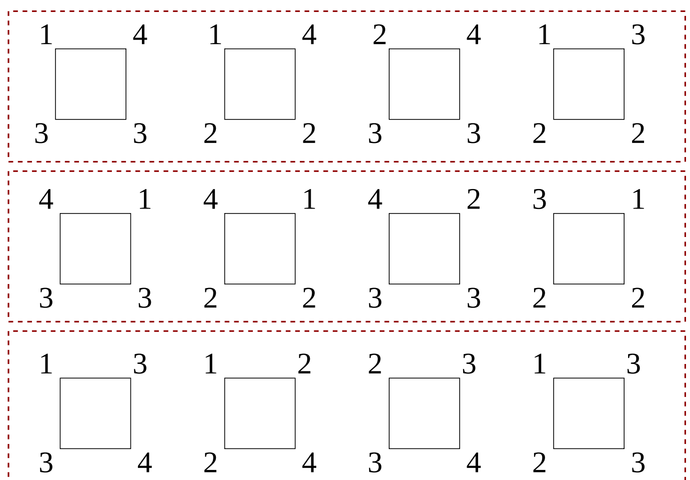
$$\mathrm{PatInv}_G(x) = \cdots + 3x_1x_3^2x_4 + \cdots .$$



M_α-expansion of PatInv_G

Let $M_{121} = \sum_{i<j<k} x_i^1 x_j^2 x_k^1$,

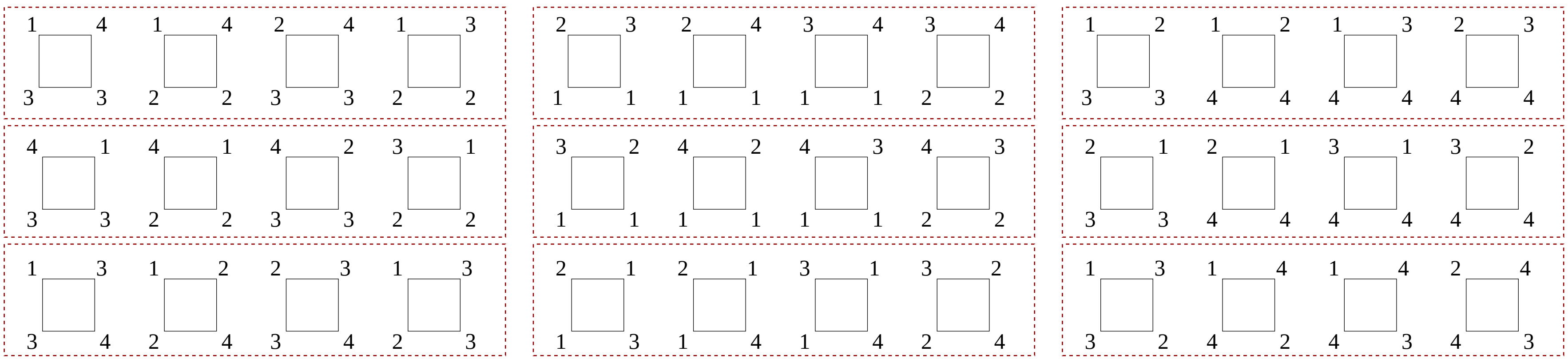
then $\mathrm{PatInv}_G(x) = \cdots + 3M_{121} + \cdots$.



m_λ-expansion of PatInv_G

Let $m_{211} = M_{211} + M_{121} + M_{112}$,

then $\mathrm{PatInv}_G(x) = \cdots + 3m_{211} + \cdots$.



M_{121} M_{211} M_{112}

Q_α-expansion of PatInv_G

Define $\boldsymbol{Q_\alpha = \sum_{\beta \text{ finer than } \alpha} M_\beta}$.

$$\mathbf{PatInv}_{C_3} = \sum_i x^i + \sum_{i<j}(x_i^2 x_j + x_i x_j^2) + 2\sum_{i<j<k} x_i x_j x_k$$

$$= M_3 + M_{21} + M_{12} + 2M_{111}$$

$$= m_3 + m_{21} + 2m_{111}$$

$$= Q_3 + Q_{111}.$$

Standardization

The standardization of a word $\boldsymbol{w}$, $\mathbf{Std}(\boldsymbol{w}) = \boldsymbol{\sigma}$, is obtained by replacing (from left to right) repeated letters by successively increasing ones.

Examples:

$$\mathbf{Std}(\mathbf{11111}) = \mathbf{12345}$$

$$\mathbf{Std}(\mathbf{12121}) = \mathbf{14253}$$

$$\mathbf{Std}(\mathbf{311421223}) = \mathbf{712943568}$$

Inverting

Fact: Standardization is invertible if you know the content of the original word.

Examples:

$$(\mathbf{12345}, \mathbf{11111})^{-1} = \mathbf{11111}$$

$$(\mathbf{12345}, \mathbf{11233})^{-1} = \mathbf{11233}$$

$$(\mathbf{14253}, \mathbf{11122})^{-1} = \mathbf{12121}$$

$$(\mathbf{14253}, \mathbf{12234})^{-1} = \mathbf{12242}$$

$$(\mathbf{712943568}, \mathbf{111222334})^{-1} = \mathbf{311421223}$$

The Standardization Bijection

Define the inverse descent set

$$\mathrm{IDes}(\boldsymbol{\sigma}) = \{j : \boldsymbol{\sigma}^{-1}(j+1) < \boldsymbol{\sigma}^{-1}(j)\} \text{ and}$$

$$\boldsymbol{W}_{n,\mathrm{IDes}(\boldsymbol{\sigma})} = \{w_1 \le w_2 \le \cdots \le w_n : \; i \in \mathrm{IDes}(\boldsymbol{\sigma}) \text{ implies } z_i < z_{i+1}\}.$$

Fact Standardization is a bijection between words of length n and pairs $(\boldsymbol{\sigma}, z)$ where $\boldsymbol{\sigma} \in S_n$ and $z \in \boldsymbol{W}_{n,\mathrm{IDes}(\boldsymbol{\sigma})}$.

Strategy

1. Define $W^+ \subset \{f : X \to C\}$ of distinguished representatives of the orbits $\mathcal{O}$. Then

$$\mathrm{PatInv}_G(x) = \sum_{\mathcal{O} \in \{f:X\to C\}/G} x_{\mathrm{wt}(\mathcal{O})} = \sum_{v \in W^+} x_{\mathrm{wt}(v)}.$$

2. Define $\mathrm{Std}' : W^+ \to S_n$. Let $S^+ = \mathrm{Std}'(W^+)$.

3. For $\sigma \in S^+$, define $\mathrm{IDes}'(\sigma) \subseteq [n-1]$.

4. Set $\mathrm{PatInv}_G = \sum_{\sigma \in S^+} Q_{\alpha(\mathrm{IDes}'(\sigma))}$.

$$G = \{e\} \leq S_n$$

Let $W = \{w_1 \cdots w_n : w_i \in C\}$.

Set $W^+ = W$ and

$\mathrm{Std}' = \mathrm{Std}$, then

$S^+ = S_n$.

Fact: $\mathrm{PatInv}_{\{e\}} = \sum_{\sigma \in S_n} Q_{\alpha(\mathrm{IDes}(\sigma))}$.

$$G = S_n \leq S_n$$

Set $W^+ = \{w_1 \leq w_2 \leq \cdots \leq w_n\}$ and

$\mathbf{Std}' = \mathbf{Std}$, then

$$S^+ = \{12\cdots n\}.$$

Prop: $\mathbf{PatInv}_{S_n} = Q_{\alpha(\mathrm{IDes}(12\cdots n))}$

$$= Q_{\alpha(\emptyset)} = h_n.$$

$G = A_n \leq S_n$

Set $W^+ = \{w_1 \leq w_2 \leq \cdots \leq w_n\} \cup$

$$\{v_1 \cdots v_n : \ v_2 < v_1 < v_3 < \cdots < v_n\}.$$

For $\mathrm{Std}' = \mathrm{Std}, S^+ = \{12\cdots n, 2134\cdots n\}.$

Set $\mathrm{IDes}'(12\cdots n) = \emptyset$ and

$$\mathrm{IDes}'(2134\cdots n) = \{1, 2, \ldots, n-1\}.$$

Prop: $\mathrm{PatInv}_{A_n} = Q_{\alpha(\mathrm{IDes}'(12\cdots n))} + Q_{\alpha(\mathrm{IDes}'(2134\cdots n))}$

$$= Q_n + Q_{1^n}.$$

Conjectures/Theorems

Perhaps $\mathbf{PatInv}_G = \sum_{\sigma \in H} Q_{\alpha(\mathrm{IDes}'(\sigma))}$ for

G acting on $[n]$	H	Proven
A_n	Z_2	Y
S_{n-1}	C_n	Y
$\langle (1,n)(2,n-1)\cdots \rangle$	A_n	M
A_{n-1}	D_{2n}	N

Scenarios

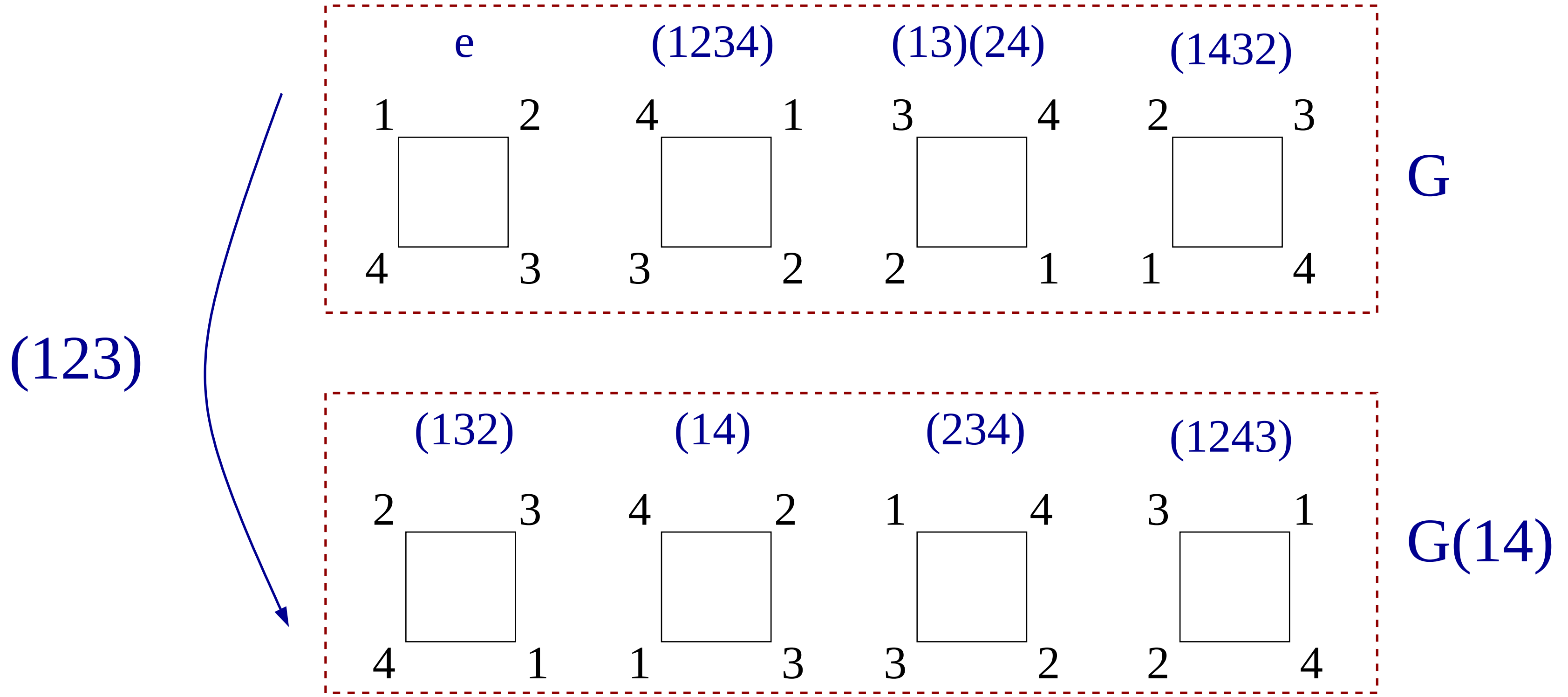
	W^+	
	transversal	other
$\mathbf{IDes}$	$\boldsymbol{S_n}$	$\boldsymbol{\langle (1,n)(2,n-1)\cdots \rangle}$
$\mathbf{IDes}'$	$\boldsymbol{\langle (1,2) \rangle}$	$\boldsymbol{C_n}$

Cycle Index

$$\mathbf{CycInd}_{G} = \frac{1}{|G|} \sum_{g \in G} p_{\rho(g)}.$$

Theorem (Pólya): $\mathbf{CycInd}_{G} = \mathbf{PatInv}_{G}.$

Permutation action



$$(1,2,3)\cdot G = G\,(1,2,3)^{-1} = G\,(1,3,2) = G\,(1,4).$$

Permutation action

$$\chi^{C_4} = \mathbf{1} \oplus \chi_{\psi} \oplus \chi_{\text{std}}.$$

The 2-D irrep of S_4, ψ, appears as:

$$\langle G + G(13) - G(34) - G(12),$$
$$G(34) + G(12) - G(23) - G(14)\rangle$$

$G = \langle (123) \rangle \leq S_n$

Set $\boldsymbol{W}^+ = \{\boldsymbol{w_1 \cdots w_n} : \ (\boldsymbol{w_1 < w_2, w_3})$ or

$\boldsymbol{w_1 = w_2 \leq w_3}\}.$

For $\mathbf{Std}' = \mathbf{Std}, \boldsymbol{S}^+ = \{\boldsymbol{\sigma : \sigma_1 < \sigma_2, \sigma_3}\}.$

Set $\mathbf{IDes}'(\boldsymbol{\sigma}) = \begin{cases} \mathbf{IDes}(\boldsymbol{\sigma}) \cup \{\boldsymbol{\sigma_1}\}, & \boldsymbol{\sigma_3 = \sigma_1 + 1}, \\ \mathbf{IDes}(\boldsymbol{\sigma}), & \text{else.} \end{cases}$

Prop: $\mathbf{PatInv}_{\langle (123) \rangle} = \sum_{\boldsymbol{\sigma \in S^+}} \boldsymbol{Q}_{\alpha(\mathbf{IDes}'(\boldsymbol{\sigma}))}.$