A Continuous Family of Partition Statistics Equidistributed with Length

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July 8, 2008

MSC Subject Classifications: 05A15, 05A17, 11P81, 05C45

Keywords: partition, arm, leg, lattice path, Eulerian tour, multigraph, bijection

Abstract

This article investigates a remarkable generalization of the generating function that enumerates partitions by area and number of parts. This generating function is given by the infinite product $\prod_{i\geq 1} 1/(1-tq^i)$. We give uncountably many new combinatorial interpretations of this infinite product involving partition statistics that arose originally in the context of Hilbert schemes. We construct explicit bijections proving that all of these statistics are equidistributed with the length statistic on partitions of n. Our bijections employ various combinatorial constructions involving cylindrical lattice paths, Eulerian tours on directed multigraphs, and oriented trees.

1 Introduction

We begin by recalling one of the most famous classical results in the theory of partitions. A *partition* of an integer $n \geq 0$ is a weakly decreasing sequence of positive integers whose sum is n. Given a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t)$, the *area* of λ is $|\lambda| = \lambda_1 + \cdots + \lambda_t$. The *length* of λ is $\ell(\lambda) = t$, the number of nonzero parts in λ. The *diagram* of λ is the set

$$\mathrm{dg}(\lambda) = \{(i,j) \in \mathbb{N}\times\mathbb{N} : 1 \leq i \leq \ell(\lambda),\ 1 \leq j \leq \lambda_i\}.$$

We visualize $\mathrm{dg}(\lambda)$ as an array of $\ell(\lambda)$ rows of boxes, left-justified, with λ_i boxes in the i'th row from the top. The *transpose* of λ is the partition λ' whose diagram is $\{(j,i) : (i,j) \in \mathrm{dg}(\lambda)\}$, so that λ'_j is the number of boxes in the j'th column of $\mathrm{dg}(\lambda)$. Let $\mathrm{Par}(n)$ denote the set of partitions of n, and let Par denote the set of all partitions.

Theorem 1.

$$\sum_{\lambda\in\mathrm{Par}} q^{|\lambda|}t^{\ell(\lambda)} = \prod_{i=1}^{\infty}\frac{1}{1-tq^i} = \sum_{\mu\in\mathrm{Par}} q^{|\mu|}t^{\mu_1}. \tag{1}$$

Proof. The infinite product appearing in the theorem can be written

$$\prod_{i=1}^{\infty}\sum_{k_i=0}^{\infty} t^{k_i}q^{ik_i}$$

by expanding $1/(1-tq^i)$ as a formal geometric series in $\mathbb{Q}[[q,t]]$. We obtain a typical term in the infinite product by choosing a monomial $t^{k_i}q^{ik_i}$ from each factor and multiplying these monomials together. Such a choice of monomials is uniquely encoded by a partition λ consisting of k_i parts equal to i, for each $i \geq 1$. Clearly $\prod_i t^{k_i}q^{ik_i} = q^{|\lambda|}t^{\ell(\lambda)}$. Adding up all these terms gives the first equation in the theorem. Setting $\mu = \lambda'$ and noting that $|\mu| = |\lambda|$ and $\mu_1 = \ell(\lambda)$, we obtain the second part of the theorem. □

This paper investigates a surprising generalization of this result, which we now describe. For each positive real number x, we will introduce two statistics on partitions, denoted h_x^+ and h_x^-. First we need some preliminary definitions. Given a partition λ and a cell $c = (i,j) \in \mathrm{dg}(\lambda)$, the *arm* of c is $a(c) = \lambda_i - j$, which is the number of cells to the right of c in its row. The *leg* of c is $l(c) = \lambda'_j - i$, which is the number of cells below c in its column. For any logical statement P, let $\chi(P) = 1$ if P is true and $\chi(P) = 0$ if P is false. For each real x such that $0 \leq x < \infty$, define

$$h_x^+(\lambda) = \sum_{c\in\mathrm{dg}(\lambda)} \chi\left(\frac{a(c)}{l(c)+1} \leq x < \frac{a(c)+1}{l(c)}\right) \qquad (\lambda \in \mathrm{Par}).$$

For all x such that $0 < x \leq \infty$, define

$$h_x^-(\lambda) = \sum_{c\in\mathrm{dg}(\lambda)} \chi\left(\frac{a(c)}{l(c)+1} < x \leq \frac{a(c)+1}{l(c)}\right) \qquad (\lambda \in \mathrm{Par}).$$

In these formulas, a fraction with a zero denominator is interpreted as $+\infty$.

Example 2. If $\lambda = (4,2,2)$, then $\mathrm{h}_\pi^+(\lambda) = 4$, $\mathrm{h}_1^+(\lambda) = 7$, $\mathrm{h}_1^-(\lambda) = 5$, $\mathrm{h}_{0.6}^-(\lambda) = 4$, $\mathrm{h}_0^+(\lambda) = 3 = \ell(\lambda)$, and $\mathrm{h}_\infty^-(\lambda) = 4 = \lambda_1$.

Note that a cell $c \in \mathrm{dg}(\lambda)$ contributes to $h_0^+(\lambda)$ iff $a(c) = 0$ iff c is the rightmost cell in its row. The number of such cells is $\ell(\lambda)$, so $h_0^+(\lambda) = \ell(\lambda)$. Similarly, c contributes to $h_\infty^-(\lambda)$ iff $l(c) = 0$ iff c is the lowest cell in its column. The number of such cells is λ_1, so $h_\infty^-(\lambda) = \lambda_1$. More generally, note that every cell $c = (i,j) \in \mathrm{dg}(\lambda)$ has an associated cell $c' = (j,i) \in \mathrm{dg}(\lambda')$ which satisfies $a(c') = l(c)$ and $l(c') = a(c)$. It follows that $h_x^\pm(\lambda) = h_{1/x}^\mp(\lambda')$ for all x and all λ.

We can now state the generalized partition theorem.

Theorem 3. *For all real* $x \in [0,\infty)$,

$$\sum_{\lambda\in\mathrm{Par}} q^{|\lambda|}t^{h_x^+(\lambda)} = \prod_{i=1}^{\infty} \frac{1}{1-tq^i}. \tag{2}$$

For all $x \in (0,\infty]$,

$$\sum_{\lambda\in\mathrm{Par}} q^{|\lambda|}t^{h_x^-(\lambda)} = \prod_{i=1}^{\infty} \frac{1}{1-tq^i}. \tag{3}$$

The classical Theorem 1 corresponds to the cases $x = 0$ and $x = \infty$ in this theorem.

Theorem 3 (for x irrational) seems to have been first discovered by Mark Haiman, although it does not appear explicitly in the literature [8]. Haiman found the following geometric proof of the theorem using results of Ellingsrud and Strømme on the Hilbert scheme of points in the plane [2, 3]. Ellingsrud and Strømme gave explicit descriptions of the Białynicki-Birula cells [1] associated to the action of a two-dimensional complex torus $T = (\mathbb{C}^*)^2$ on the Hilbert scheme. In [9], Haiman explicitly computed a system of local parameters at each T-fixed point that are T-eigenfunctions and are nicely indexed combinatorially. The choice of the "slope" parameter $x \notin \mathbb{Q}$ corresponds to the choice of a one-parameter torus in T. The dimensions of the Białynicki-Birula cells for this one-parameter torus are then distributed according to the partition statistic h_x^+ (which equals h_x^- for x irrational). On the other hand, these dimensions are always distributed according to the Betti numbers of the Hilbert scheme. Thus the distribution of h_x^+ is independent of x. This completes Haiman's geometric proof of the theorem [8]. Some related work involving the statistic h_1^+ may be found in [10, 11].

Our goal in this document is to give a purely combinatorial proof of Theorem 3, using no algebraic geometry. The main steps in our proof are as follows:

1. We show that Theorem 3 is a consequence of the following result.

   Theorem 4. *For all positive* rational *numbers x and all integers $n \geq 0$,*

$$\sum_{\lambda \in \mathrm{Par}(n)} t^{h_x^+(\lambda)} = \sum_{\lambda \in \mathrm{Par}(n)} t^{h_x^-(\lambda)}. \tag{4}$$

2. We fix a positive rational number $x = r/s$ and define new statistics mid_x, c_x^+, and c_x^- on partitions. These statistics have the property that $\mathrm{mid}_x + \mathrm{c}_x^+ = \mathrm{h}_x^+$ and $\mathrm{mid}_x + \mathrm{c}_x^- = \mathrm{h}_x^-$. We will consider generating functions

$$F_{r,s,n}(q,z,w,y) = \sum_{\lambda} q^{|\lambda|} z^{\mathrm{mid}_{r/s}(\lambda)} w^{\mathrm{c}_{r/s}^+(\lambda)} y^{\mathrm{c}_{r/s}^-(\lambda)}$$

   where the sum extends over partitions λ contained in a right triangle of size rn by sn. We will see that Theorem 4 is implied by the following symmetry property.

   Theorem 5. *Suppose r and s are relatively prime positive integers and $n \geq 0$. Then*

$$F_{r,s,n}(q,z,w,y) = F_{r,s,n}(q,z,y,w). \tag{5}$$

3. We give a bijective proof of Theorem 5. The first step is to associate to each partition λ a certain directed multigraph $M(\lambda)$ and Eulerian tour $\mathcal{E}(\lambda)$, following a construction of Jonas Sjöstrand [13]. We show that the statistics $|\lambda|$, $\mathrm{mid}_{r/s}(\lambda)$, and $\mathrm{c}^+(\lambda) + \mathrm{c}^-(\lambda)$ depend only on the multigraph $M(\lambda)$, not on the Eulerian tour $\mathcal{E}(\lambda)$. We then construct an involution that modifies the Eulerian tour $\mathcal{E}(\lambda)$ in such a way that the statistics c^+ and c^- are interchanged. This induces a map on partitions that switches c^+ and c^- while fixing the area and mid statistics. The well-known connection between Eulerian tours and oriented trees (cf. §5.6 of [14]) plays a key role in constructing these maps.

Our proof of Theorem 3 will be completely bijective. More precisely, for any $x, y \in [0, \infty]$, any $\delta, \epsilon \in \{+, -\}$, and any $n \geq 0$, we will construct an explicit bijection on $\mathrm{Par}(n)$ that sends the statistic h_x^δ to the statistic h_y^ϵ. (Here and below, we exclude the two choices $(x, \delta) = (0, -)$ and $(x, \delta) = (\infty, +)$.) This bijection is essentially a composition of finitely many bijections that switch h_r^+ and h_r^- at each "critical" rational number r between x and y. (This terminology is explained in the next section.) The net result is a kind of "combinatorial homotopy" that slowly deforms the original partition into its image as the parameter value goes from x to y. See Figure 6 for an example.

The rest of the paper is organized as follows. We show that Theorem 4 implies Theorem 3 in Section 2. We show that Theorem 5 implies Theorem 4 in Section 3. Section 4 describes various combinatorial encodings of partitions in terms of lattice paths, Eulerian tours on directed multigraphs, and indexed collections of binary words. Section 5 derives new formulas for our partition statistics in terms of these encodings. Section 6 defines the involution used to prove Theorem 5. Section 6.3 contains an example of the "combinatorial homotopy" mentioned above. Section 7 concludes by describing an intriguing link between Theorem 3 and an unsolved problem involving the Garsia-Haiman q,t-Catalan numbers.

2 Reduction to Critical Rationals

Proposition 6. *Theorem 4 implies Theorem 3.*

Proof. For $x \in [0,\infty]$ and $\delta \in \{+,-\}$, define

$$H_x^{\delta}(n) = \sum_{\lambda\in\mathrm{Par}(n)} t^{h_x^{\delta}(\lambda)} \in \mathbb{N}[t].$$

We will show that $H_x^{\delta}(n)$ is independent of x and δ. In particular, this yields

$$H_x^{\delta}(n) = H_0^{+}(n) = \sum_{\lambda\in\mathrm{Par}(n)} t^{\ell(\lambda)}$$

for all n, x, and δ. Theorem 3 follows immediately by multiplying by q^n, adding over all $n \geq 0$, and applying Theorem 1.

Fix an integer $n \geq 0$. We say that a positive rational number r is a *critical rational for* n iff there exists a partition $\mu \in \mathrm{Par}(n)$ and a cell $c \in \mathrm{dg}(\mu)$ such that $\frac{a(c)}{l(c)+1} = r$ or $\frac{a(c)+1}{l(c)} = r$. By convention, 0 and $+\infty$ are also considered to be critical rationals for every n. Let $\mathrm{Crit}(n)$ denote the set of critical rational numbers for n; evidently, $\mathrm{Crit}(n)$ is finite. For example,

$$\mathrm{Crit}(5) = \{0, 1/4, 1/3, 1/2, 2/3, 1, 3/2, 2, 3, 4, +\infty\}.$$

(More generally, it is easy to check that $\mathrm{Crit}(0) = \{0,\infty\}$ and $\mathrm{Crit}(n) = \mathrm{Crit}(n-1) \cup \{a/(n-a) : 0 < a < n\}$ for $n \geq 1$.) Write $\mathrm{Crit}(n) = \{0 = r_0 < r_1 < \cdots < r_k = +\infty\}$, where k depends on n. Define open intervals $I_j = (r_{j-1}, r_j)$ for $1 \leq j \leq k$. Then $[0,\infty]$ decomposes into the disjoint union

$$[0,\infty] = I_1 \cup I_2 \cup \cdots \cup I_k \cup \mathrm{Crit}(n).$$

Let x, x' be two elements of the same interval I_j, and let $\delta, \delta' \in \{+,-\}$. Suppose λ is any partition of n. Since there are no critical rational numbers between x and x' (inclusive), it is immediate from the definitions of the statistics that a cell $c \in \mathrm{dg}(\lambda)$ contributes to $h_x^{\delta}(\lambda)$ iff c contributes to $h_{x'}^{\delta'}(\lambda)$. Thus $t^{h_x^{\delta}(\lambda)} = t^{h_{x'}^{\delta'}(\lambda)}$. Adding over all λ, we see that for all $x, x' \in I_j$,

$$H_x^{\delta}(n) = H_{x'}^{\delta'}(n) \in \mathbb{N}[t]. \tag{6}$$

A similar argument shows that for all $x \in I_j$,

$$H_{r_{j-1}}^{+}(n) = H_x^{\delta} = H_{r_j}^{-}(n) \quad \forall n \geq 0. \tag{7}$$

On the other hand, the assumed equations (4) imply in particular that

$$H_{r_j}^{+}(n) = H_{r_j}^{-}(n) \quad (n \geq 0,\ 1 \leq j \leq k-1). \tag{8}$$

Equations (6), (7), and (8) clearly imply that $H_x^{\delta}(n)$ is independent of x and δ. □

Suppose $x < x'$ and $\delta, \delta' \in \{+,-\}$ are given. Suppose further that we have bijective proofs of (8) for each critical rational r_j. Then we can construct a bijective proof of the identity $H_x^{\delta}(n) = H_{x'}^{\delta'}(n)$ by simply chaining together the bijections used at each critical rational lying between x and x'. (An example of this process is given in §6.3.) Thus a bijective proof of Theorem 4 at all critical rationals leads immediately to a bijective proof of Theorem 3.

3 Reduction to Symmetry Property

In this section, we prove that Theorem 5 implies Theorem 4. We must first define the partition statistics mid_x, c_x^+, and c_x^- and the generating function $F_{r,s,n}$ appearing in the statement of Theorem 5. Given a positive rational number x, write $x = r/s$ where r and s are positive integers with $\gcd(r,s) = 1$. For each $\lambda \in \mathrm{Par}$, define the *middle* statistic, the *critical-plus* statistic, the *critical-minus* statistic, and the *critical-total* statistic for λ as follows:

$$\begin{aligned}
\mathrm{mid}_{r/s}(\lambda) &= \sum_{c\in\mathrm{dg}(\lambda)} \chi(sa(c) - rl(c) \in (-s, +r)) \\
\mathrm{c}_{r/s}^{+}(\lambda) &= \sum_{c\in\mathrm{dg}(\lambda)} \chi(sa(c) - rl(c) = +r) \\
\mathrm{c}_{r/s}^{-}(\lambda) &= \sum_{c\in\mathrm{dg}(\lambda)} \chi(sa(c) - rl(c) = -s) \\
\mathrm{ctot}_{r/s}(\lambda) &= \mathrm{c}_{r/s}^{+}(\lambda) + \mathrm{c}_{r/s}^{-}(\lambda).
\end{aligned}$$

By setting $x = r/s$ in the definitions of h_x^+ and h_x^- and clearing fractions, we see that

$$\mathrm{h}_{r/s}^{+}(\lambda) = \sum_{c\in\mathrm{dg}(\lambda)} \chi(sa(c) - rl(c) \in (-s, +r]) = \mathrm{mid}_{r/s}(\lambda) + \mathrm{c}_{r/s}^{+}(\lambda); \tag{9}$$

$$\mathrm{h}_{r/s}^{-}(\lambda) = \sum_{c\in\mathrm{dg}(\lambda)} \chi(sa(c) - rl(c) \in [-s, +r)) = \mathrm{mid}_{r/s}(\lambda) + \mathrm{c}_{r/s}^{-}(\lambda). \tag{10}$$

Example 7. Let $r = 3$, $s = 2$, and $\lambda = (12, 12, 10, 8, 7, 4, 1, 1, 1)$. Then $|\lambda| = 56$, $\mathrm{mid}_{3/2}(\lambda) = 29$, $\mathrm{c}_{3/2}^{+}(\lambda) = 6$, $\mathrm{c}_{3/2}^{-}(\lambda) = 3$, $\mathrm{ctot}_{3/2}(\lambda) = 9$, $\mathrm{h}_{3/2}^{+}(\lambda) = 35$, $\mathrm{h}_{3/2}^{-}(\lambda) = 32$.

Next we define $\mathrm{Par}_{r,s,n}$ and $F_{r,s,n}$. For $\mu \in \mathrm{Par}$, let $\mathrm{dg}_{r,s,n}(\mu)$ be the diagram of μ (regarded as a collection of unit squares in $\mathbb{R}^2$) translated so that the northwest corner of the northwesternmost cell has x, y-coordinates $(0, sn)$. Let $\Delta_{r,s,n}$ be the closed triangle in $\mathbb{R}^2$ with vertices $(0, 0)$, $(0, sn)$, and (rn, sn). Define

$$\mathrm{Par}_{r,s,n} = \{\mu \in \mathrm{Par} : \mathrm{dg}_{r,s,n}(\mu) \subseteq \Delta_{r,s,n}\}.$$

For example, Figure 1 shows that $(12, 12, 10, 8, 7, 4, 1, 1, 1) \in \mathrm{Par}_{3,2,5}$. Finally, define

$$F_{r,s,n}(q, z, w, y) = \sum_{\lambda\in\mathrm{Par}_{r,s,n}} q^{|\lambda|} z^{\mathrm{mid}_{r/s}(\lambda)} w^{\mathrm{c}_{r/s}^{+}(\lambda)} y^{\mathrm{c}_{r/s}^{-}(\lambda)}.$$

Theorem 5 asserts that $F_{r,s,n}(q, z, w, y) = F_{r,s,n}(q, z, y, w)$.

Proposition 8. *Theorem 5 implies Theorem 4.*

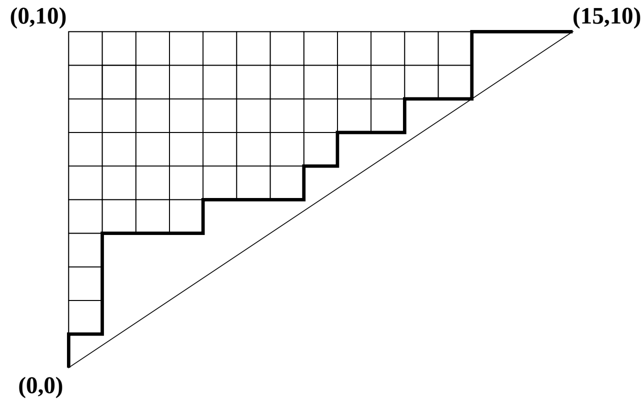

Figure 1: A partition diagram inscribed in a triangle.

Proof. We must prove that h_x^+ and h_x^- are equidistributed on $\mathrm{Par}(m)$ for all positive rational x and all integers $m \geq 0$. Suppose $x = r/s > 0$ and $m \geq 0$ are given. Choose n large enough that $\mathrm{Par}(m) \subseteq \mathrm{Par}_{r,s,n}$. (For instance, it suffices to choose $n \geq (r+s)m/rs$.) In the assumed equation $F_{r,s,n}(q,z,w,y) = F_{r,s,n}(q,z,y,w)$, set $z = wy$ and extract the coefficient of q^m. Using (9) and (10), we obtain the formula

$$\sum_{\lambda \in \mathrm{Par}(m)} w^{h^+_{r/s}(\lambda)} y^{h^-_{r/s}(\lambda)} = \sum_{\lambda \in \mathrm{Par}(m)} w^{h^-_{r/s}(\lambda)} y^{h^+_{r/s}(\lambda)}.$$

To complete the proof, just set $w = t$ and $y = 1$. □

Thus all the theorems in the introduction follow from Theorem 5. Indeed, Theorem 5 is much stronger than the others since it shows that $h^+_{r/s}$ and $h^-_{r/s}$ are *jointly* symmetric on $\mathrm{Par}(m)$, and it also gives information about the 4-variate distribution of area, $\mathrm{mid}_{r/s}$, $\mathrm{c}^+_{r/s}$, and $\mathrm{c}^-_{r/s}$ on the collection of partitions contained in any $rn \times sn$ triangle. In the coming sections, we will prove Theorem 5 by constructing involutions on the sets $\mathrm{Par}_{r,s,n}$ that preserve area and $\mathrm{mid}_{r/s}$ while interchanging $\mathrm{c}^+_{r/s}$ and $\mathrm{c}^-_{r/s}$. We will also prove explicit fermionic formulas giving the joint distribution of the four statistics on these collections of partitions.

4 Combinatorial Descriptions of Partitions

To prove Theorem 5, it will be helpful to introduce ways of encoding partitions in $\mathrm{Par}_{r,s,n}$ using lattice paths, Eulerian tours on directed multigraphs, and families of binary words. We discuss these encodings in this section. Our immediate goal is to prove the structural results given below in Theorem 14 and its corollary. Throughout this section, we fix positive integers r, s, n such that $\gcd(r,s) = 1$.

4.1 Preliminary Definitions

A *word* is an ordered sequence of letters drawn from some alphabet. Let $W(E^aN^b)$ be the set of all words that consist of a copies of the letter E and b copies of the letter N. We can view words $w \in W(E^aN^b)$ as *lattice paths* from $(0,0)$ to (a,b) by interpreting E as a unit east step and N as a unit north step. A word

$w \in W(E^{rn}N^{sn})$ is called an *r/s-Dyck word of order n* iff every point (x, y) on the associated lattice path satisfies $y \geq (s/r)x$. This means that the lattice path lies completely within the triangle $\Delta_{r,s,n}$.

A *directed graph* is an ordered pair $G = (V_G, E_G)$, where V_G is a set of *vertices* and $E_G \subseteq V_G \times V_G$ is a set of *(directed) edges*. Given an edge $e = (v, w)$, we set $\mathrm{init}(e) = v$ and $\mathrm{fin}(e) = w$. A *multigraph* is a pair $M = (V_M, E_M)$, where V_M is a set of vertices and E_M is now a *multiset* of directed edges. This means that each edge occurs in E_M with a certain multiplicity. Graphs are special kinds of multigraphs in which each edge has multiplicity one. The *indegree* of a vertex $v \in V_M$, denoted $\mathrm{indeg}(v)$, is the number of edges $e \in E_M$ such that $\mathrm{fin}(e) = v$ (counted with multiplicities). The *outdegree* of a vertex $v \in V_M$, denoted $\mathrm{outdeg}(v)$, is the number of edges $e \in E_M$ such that $\mathrm{init}(e) = v$. A multigraph M is *balanced* iff $\mathrm{indeg}(v) = \mathrm{outdeg}(v)$ for all $v \in V_M$. An *isolated vertex* of M is a vertex $v \in V_M$ such that $\mathrm{init}(e) \neq v \neq \mathrm{fin}(e)$ for every edge e.

For $k \geq 1$, a *trail of length* k in $M = (V_M, E_M)$ is a sequence $P = (v_0, v_1, \ldots, v_k)$ such that each $v_i \in V_M$ and $(v_{i-1}, v_i) \in E_M$ for $1 \leq i \leq k$. We say that the trail *starts* at v_0 and *ends* at v_k. A trail is *closed* iff $v_0 = v_k$. A *path* is a trail in which all vertices are distinct, except that we allow $v_0 = v_k$. A *cycle* is a closed path. The *edge multiset* of a trail is the multiset $E(P) = \{(v_{i-1}, v_i) : 1 \leq i \leq k\}$. An *Eulerian tour* of M is a closed trail in M whose edge multiset is precisely E_M. An *oriented tree* in M *leading from the root* v_0 is a graph $T = (V_T, E_T)$ such that $V_T \subseteq V_M$, E_T is a subset of the multiset E_M, $v_0 \in V_T$, and for each $v \neq v_0$ in V_T there exists a unique path in T from v_0 to v. We write $\mathrm{dist}_T(v_0, v)$ to denote the length of this unique path. The tree T is said to *span* M iff $V_T = V_M$. Oriented trees leading *to* the root v_0 are defined analogously. A multigraph M is *connected* iff for any two distinct vertices $v, w \in V_M$, there exists a path in M from v to w. It is well-known that a multigraph M with no isolated vertices has an Eulerian tour iff M is connected and balanced [14].

4.2 Lattice Path Formulation

We can represent a partition $\mu \in \mathrm{Par}_{r,s,n}$ as an r/s-Dyck path of order n by "following the frontier of μ." More specifically, inscribe the diagram of μ in the triangle $\Delta_{r,s,n}$ as shown in Figure 1. Define the lattice path $\mathrm{Bdy}(\mu) \in W(E^{rn}N^{sn})$ by taking north and east steps from $(0, 0)$ to (rn, sn) along the southeast boundary of $\mathrm{dg}_{r,s,n}(\mu)$. For example, the boundary of the partition $\mu = (12, 12, 10, 8, 7, 4, 1, 1, 1) \in \mathrm{Par}_{3,2,5}$ is shown as a thick shaded line in Figure 1. We have

$$\mathrm{Bdy}(\mu) = \text{NENNNEEENEEENENEENEENNEEE} \in W(E^{15}N^{10}).$$

Define $\mathrm{VBdy}(\mu)$ to be the sequence of lattice points in $\mathbb{R}^2$ visited by the path $\mathrm{Bdy}(\mu)$. More explicitly,

$$\mathrm{VBdy}(\mu) = ((x_0, y_0), (x_1, y_1), \ldots, (x_{rn+sn}, y_{rn+sn}))$$

where $(x_0, y_0) = (0, 0)$, $(x_i, y_i) = (x_{i-1} + 1, y_{i-1})$ if $\mathrm{Bdy}(\mu)_i = E$, and $(x_i, y_i) = (x_{i-1}, y_{i-1} + 1)$ if $\mathrm{Bdy}(\mu)_i = N$. Note that $(x_{rn+sn}, y_{rn+sn}) = (rn, sn)$, and $y_i \geq (s/r)x_i$ for all i since $\mathrm{dg}_{r,s,n}(\mu) \subseteq \Delta_{r,s,n}$. Clearly, μ is uniquely recoverable from either $\mathrm{Bdy}(\mu)$ or $\mathrm{VBdy}(\mu)$.

4.3 Eulerian Tour Formulation

The next step is to encode each partition $\mu \in \mathrm{Par}_{r,s,n}$ as an Eulerian tour $\mathcal{E}(\mu)$ on a certain multigraph $M(\mu) = (V_M(\mu), E_M(\mu))$, following a construction of Jonas Sjöstrand [13]. To define these objects, we first introduce the *r/s-diagonal map* $d_{r,s} : \mathbb{R}^2 \to \mathbb{R}$ given by $d_{r,s}(x, y) = ry - sx$. Write $\mathrm{VBdy}(\mu) = ((x_0, y_0), \ldots, (x_{rn+sn}, y_{rn+sn}))$ as above. Now define $\mathcal{E}(\mu) = (v_0, v_1, \ldots, v_{rn+sn})$, where $v_i = d_{r,s}(x_i, y_i)$. (By the definition of $\mathrm{VBdy}(\mu)$, it is equivalent to set $v_0 = 0$, $v_i = v_{i-1} + r$ if

$\mathrm{Bdy}(\mu)_i = N$, and $v_i = v_{i-1} - s$ if $\mathrm{Bdy}(\mu)_i = E$.) Since $\mathrm{Bdy}(\mu)$ is an r/s-Dyck path, it follows that $v_0 = v_{rn+sn} = 0$ and $v_i \geq 0$ for all i. Finally, define the multigraph $M(\mu)$ by setting $V_M(\mu) = \{v_i : 0 \leq i \leq rn + sn\}$ and letting $E_M(\mu)$ be the edge multiset $E(\mathcal{E}(\mu)) = \{(v_{i-1}, v_i) : 1 \leq i \leq rn + sn\}$. It is automatic from this definition that $\mathcal{E}(\mu)$ is an Eulerian tour on $M(\mu)$ starting and ending at 0.

Example 9. Given $\mu = (12, 12, 10, 8, 7, 4, 1, 1, 1) \in \mathrm{Par}_{3,2,5}$. Using Figure 1, we compute:

$$\mathcal{E}(\mu) = (0, 3, 1, 4, 7, 10, 8, 6, 4, 7, 5, 3, 1, 4, 2, 5, 3, 1, 4, 2, 0, 3, 6, 4, 2, 0).$$

The multigraph $M(\mu)$ has vertex set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 10\}$ and edge multiset

$$\{(0, 3) \times 2, (1, 4) \times 3, (2, 5) \times 1, (3, 6) \times 1, (4, 7) \times 2, (7, 10) \times 1,$$

$$(2, 0) \times 2, (3, 1) \times 3, (4, 2) \times 3, (5, 3) \times 2, (6, 4) \times 2, (7, 5) \times 1, (8, 6) \times 1, (10, 8) \times 1\}.$$

The notation $(v, v') \times k$ means that the edge (v, v') occurs with multiplicity k in the given multiset.

The multigraph $M(\mu)$ has the following properties. (1) The vertex set $V_M(\mu)$ contains 0 and is a finite subset of $\mathbb{N}$. (2) $M(\mu)$ has exactly $rn + sn$ directed edges (counting multiplicities). (3) Every edge in $M(\mu)$ is either a *north edge* leading from some vertex v to $v + r$ or an *east edge* leading from v to $v - s$. (4) $M(\mu)$ is connected, balanced, and has no isolated vertices. We let $\mathrm{MGraph}_{r,s,n}$ be the set of all multigraphs having properties (1) through (4).

A convenient way to visualize the multigraph $M(\mu)$ is to draw all the vertices between the lines $x + y = 0$ and $x + y = r + s$ in $\mathbb{R}^2$, with "wraparound" at these two edges. A lattice point (x, y) in this region corresponds to the vertex $v = d_{r,s}(x, y)$ in the multigraph. In this picture, edges from v to $v + r$ are indeed "north edges" in the usual sense, while edges from v to $v - s$ are "east edges" in the usual sense. Note that each vertex in the multigraph is obtained by "collapsing" all the lattice points on the r/s-diagonal $ry - sx = v$ into the single vertex v. Thus, we can view the tour $\mathcal{E}(\mu)$ as a *cylindrical lattice path* obtained by collapsing the ordinary lattice path $\mathrm{Bdy}(\mu)$ onto an "r/s-cylinder." The multigraph from the previous example is illustrated in Figure 2.

For each multigraph $M \in \mathrm{MGraph}_{r,s,n}$, define

$$\mathrm{Par}_M = \{\mu \in \mathrm{Par}_{r,s,n} : M(\mu) = M\}.$$

Also define ETour_M to be the set of all Eulerian tours $\mathcal{E}$ on M that begin and end at vertex 0. We have just seen that every $\mu \in \mathrm{Par}_M$ has an associated Eulerian tour $\mathcal{E}(\mu) \in \mathrm{ETour}_M$. Conversely, it is clear that any Eulerian tour $T \in \mathrm{ETour}_M$ has the form $\mathcal{E}(\mu)$ for a unique partition $\mu \in \mathrm{Par}_M$. For, given $T = (0 = v_0, v_1, \ldots, v_{rn+sn})$, μ is determined by the conditions $\mathrm{Bdy}(\mu)_i = N$ if $v_i - v_{i-1} = r$ and $\mathrm{Bdy}(\mu)_i = E$ if $v_i - v_{i-1} = -s$. In summary, we have canonical bijections $\mathrm{Par}_M \to \mathrm{ETour}_M$ for each M, which assemble to give a canonical bijection

$$\mathrm{Par}_{r,s,n} \to \bigcup_{M \in \mathrm{MGraph}_{r,s,n}} \mathrm{ETour}_M\,.$$

4.4 Formulation using Arrival Words and Departure Words

We now introduce a convenient description of Eulerian tours and multigraphs using sequences of binary words. Given $\mu \in \mathrm{Par}_{r,s,n}$, define the sequence of *arrival words* $(w^v(\mu) : v \geq 0)$ as follows. Write $\mathrm{Bdy}(\mu) = u_1 \cdots u_{rn+sn}$ and $\mathrm{VBdy}(\mu) = ((x_i, y_i) : 0 \leq i \leq rn + sn)$, as usual. Given v, let

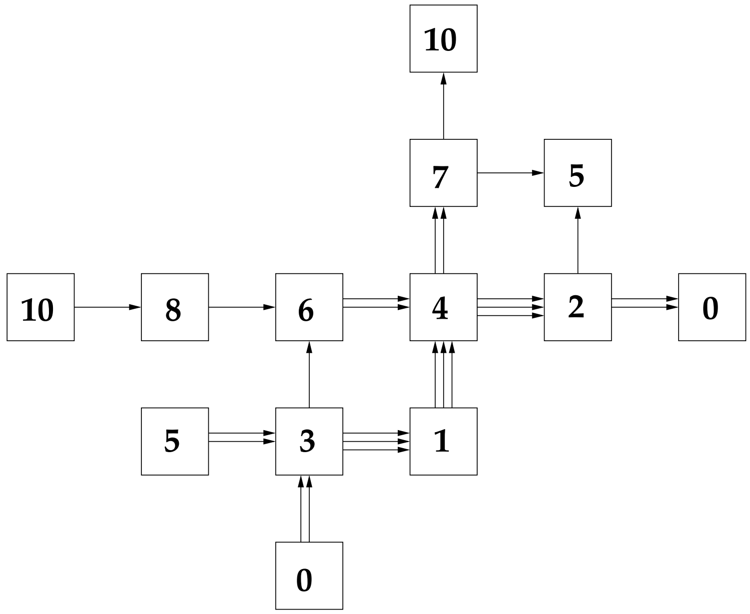

Figure 2: Multigraph associated to the partition μ of Example 9.

$j_1 < j_2 < \ldots < j_m$ be the indices such that $d_{r,s}(x_{j_k}, y_{j_k}) = v$. Define $w^v(\mu) = u_{j_1}u_{j_2}\cdots u_{j_m}$. Informally, we construct the arrival word w^v by traversing the Eulerian tour $\mathcal{E}(\mu)$, recording an N every time the tour arrives at v via a north step from vertex $v-r$, and recording an E every time the tour arrives at v via an east step from vertex $v+s$. Thus if $M(\mu)$ has a_i east edges entering v and b_i north edges entering v, we have $w^v(\mu) \in W(E^{a_i}N^{b_i})$.

Example 10. For $\mu = (12, 12, 10, 8, 7, 4, 1, 1, 1)$, the nonempty arrival words are:

$$w^0 = EE,\ w^1 = EEE,\ w^2 = EEE,\ w^3 = NEEN,\ w^4 = NENNE,$$

$$w^5 = EN,\ w^6 = EN,\ w^7 = NN,\ w^8 = E,\ w^{10} = N.$$

We can use a dual construction to define the sequence of *departure words* $(y^v(\mu) : v \geq 0)$ for $\mu \in \mathrm{Par}_{r,s,n}$. Fix v. Writing $\mathrm{Bdy}(\mu)$ and $\mathrm{VBdy}(\mu)$ as above, let $j_1 < j_2 < \ldots < j_m$ be the indices such that $d_{r,s}(x_{j_k-1}, y_{j_k-1}) = v$. Define $y^v(\mu) = u_{j_1}u_{j_2}\cdots u_{j_m}$. Informally, we construct the departure word y^v by traversing the Eulerian tour $\mathcal{E}(\mu)$, recording an N every time the tour leaves v going north to $v+r$, and recording an E every time the tour leaves v going east to $v-s$. Thus if $M(\mu)$ has c_i east edges leaving v and d_i north edges leaving v, we have $y^v(\mu) \in W(E^{c_i}N^{d_i})$.

Example 11. For $\mu = (12, 12, 10, 8, 7, 4, 1, 1, 1)$, the nonempty departure words are:

$$y^0 = NN,\ y^1 = NNN,\ y^2 = NEE,\ y^3 = EEEN,\ y^4 = NNEEE,$$

$$y^5 = EE,\ y^6 = EE,\ y^7 = NE,\ y^8 = E,\ y^{10} = E.$$

There is no loss of information in the passage from μ to $(y^v(\mu) : v \geq 0)$. For, knowing this sequence of departure words, we can first recover the numbers c_i and d_i, which are sufficient to reconstitute the multigraph $M(\mu)$. Next, we can recover the Eulerian tour $\mathcal{E}(\mu)$ (or equivalently, the lattice path $\mathrm{Bdy}(\mu)$)

by simply moving forward through the multigraph starting at vertex 0. At each vertex, we consult the next unused character in the departure word for that vertex to decide which step to take next (north or east). We continue in this way for $rn+sn$ steps. Similarly, we can recover $M(\mu)$, $\mathcal{E}(\mu)$, $\mathrm{Bdy}(\mu)$ and μ from the sequence of arrival words $(w^v(\mu) : v \geq 0)$. The only difference is that now we must reconstruct $\mathrm{Bdy}(\mu)$ going backwards from (rn, sn) to $(0,0)$. At each step, we consult the next unused character in the arrival word for the current vertex — *reading right to left* — to decide what the previous step in the path must have been.

Example 12. Suppose that $\nu \in \mathrm{Par}_{3,2,5}$ has the following sequence of arrival words.

$$w^0 = EE,\ w^1 = EEE,\ w^2 = EEE,\ w^3 = NNEE,\ w^4 = NENNE,$$
$$w^5 = NE,\ w^6 = EN,\ w^7 = NN,\ w^8 = E,\ w^{10} = N.$$

By counting E's and N's in each word, we discover that $M(\nu)$ is the multigraph shown in Figure 2. Moving backwards through the multigraph from vertex 0, we recover the sequence of steps

$$EEENEENNEENENENENEEEEENNNEN.$$

Reversing this word and drawing the resulting lattice path, we find that $\nu = (12, 10, 10, 8, 7, 6, 1, 1, 1)$.

4.5 Characterization of Valid Arrival Words

We can summarize the results of the previous subsection in the following way. Given a multigraph $M \in \mathrm{MGraph}_{r,s,n}$ and a vertex $v \in V_M$, let $E_{\mathrm{in}}(v, M)$ be the number of east edges of M entering v. The quantities $E_{\mathrm{out}}(v, M)$, $N_{\mathrm{in}}(v, M)$, and $N_{\mathrm{out}}(v, M)$ are defined similarly. Then we have shown that the "arrival map" $\mu \mapsto (w^v(\mu) : v \in V_M)$ gives an injection

$$\phi_a : \mathrm{Par}_M \to \prod_{v \in V_M} W(E^{E_{\mathrm{in}}(v,M)} N^{N_{\mathrm{in}}(v,M)}).$$

Similarly, the "departure map" $\mu \mapsto (y^v(\mu) : v \in V_M)$ gives an injection

$$\phi_d : \mathrm{Par}_M \to \prod_{v \in V_M} W(E^{E_{\mathrm{out}}(v,M)} N^{N_{\mathrm{out}}(v,M)}).$$

However, these maps are *not* surjective in general. For, suppose we take an arbitrary sequence of arrival words (resp. departure words) and perform the tour-regeneration procedure indicated in the last subsection. Then it may well happen that we arrive back at vertex 0 after fewer than $rn + sn$ steps with no unused edges left leading into (resp. away from) vertex zero. In this situation, the given sequence of words lies outside the image of ϕ_a (resp. ϕ_d).

Example 13. Consider the following sequence of arrival words.

$$w^0 = EE,\ w^1 = EEE,\ w^2 = EEE,\ w^3 = NEEN,\ w^4 = EENNN,$$
$$w^5 = NE,\ w^6 = EN,\ w^7 = NN,\ w^8 = E,\ w^{10} = N.$$

Drawing the multigraph and moving backwards from 0, we recover the sequence of steps

$$EENENEENEEENNEENEENN.$$

At this point, we have returned to vertex 0, and there are no unused letters left in w^0. Yet, we have not used all the edges in the multigraph! So the given collection of words is not in the image of ϕ_a.

We will now derive a simple characterization of the image of the map ϕ_a (a dual result holds for ϕ_d). This characterization follows readily from the connection between Eulerian tours and oriented trees given in the proof of Theorem 5.6.2 of [14]. For the convenience of the reader, we recall the relevant details now.

First we need a few more definitions. Let $W_E(E^a N^b)$ be the set of all $w \in W(E^a N^b)$ such that $w_1 = E$, and let $W_N(E^a N^b)$ be the set of all $w \in W(E^a N^b)$ such that $w_1 = N$. Dually, let $W(E^a N^b)_E$ be the set of all $w \in W(E^a N^b)$ such that $w_{a+b} = E$, and let $W(E^a N^b)_N$ be the set of all $w \in W(E^a N^b)$ such that $w_{a+b} = N$. Given a multigraph $M \in \mathrm{MGraph}_{r,s,n}$, let $\mathrm{TreeA}(M)$ be the set of all oriented spanning trees leading from the root 0. Given $T \in \mathrm{TreeA}(M)$, each nonzero vertex v in M has a unique edge of T leading into it. Write $T_v = N$ if this is a north edge, and write $T_v = E$ if this is an east edge. Dual definitions apply to the set $\mathrm{TreeD}(M)$ of all oriented spanning trees leading to the root 0.

Theorem 14. *For each $M \in \mathrm{MGraph}_{r,s,n}$, the arrival map is a bijection*

$$\phi_a : \mathrm{Par}_M \cong \bigcup_{T \in \mathrm{TreeA}(M)} \prod_{v \in V_M} W_{T_v}(E^{E_{\mathrm{in}}(v,M)} N^{N_{\mathrm{in}}(v,M)}).$$

Proof. Let S denote the set on the right side of the theorem statement. First we show that $\phi_a(\mu) \in S$ for each $\mu \in \mathrm{Par}_M$. Given μ, write $\phi_a(\mu) = (w^v(\mu) : v \in V_M)$ as usual. Define a subgraph $T = (V_T, E_T)$ of M as follows. The vertex set V_T is V_M. For each nonzero vertex v of V_M, there is exactly one edge of T leading into v. This edge is an east edge if $w^v(\mu)_1 = E$ and a north edge if $w^v(\mu)_1 = N$. Vertex zero has no edges leading into it. Note that the edges of T encode the "first arrivals" of the Eulerian tour $\mathcal{E}(\mu)$ at each nonzero vertex v. It follows from this definition that $w^v(\mu) \in W_{T_v}(E^{E_{\mathrm{in}}(v,M)} N^{N_{\mathrm{in}}(v,M)})$ for each v. Thus we need only verify that $T \in \mathrm{TreeA}(M)$.

Note that the directed graph T has $|V_M|$ vertices and $|V_M|-1$ edges. Each nonzero vertex has exactly one edge of T leading into it. To check that T is an oriented spanning tree leading from zero, it evidently suffices to prove that T has no cycles. Assume, to get a contradiction, that $C = (z_0, z_1, \ldots, z_k)$ is such a cycle, where $k > 0$ and $z_k = z_0$. Among the vertices in C, let z_i be the vertex that occurs earliest in the Eulerian tour $\mathcal{E}(\mu)$. Since $z_0 = z_k$, we can choose i so that $1 \leq i \leq k$. Since there is an edge from z_{i-1} to z_i in T, z_i cannot be vertex 0. Thus, the tour entered z_i for the first time from some other vertex z'. This vertex z' *must* be z_{i-1}, by definition of the edge set of T. But $z_{i-1} \neq z_i$, so this contradicts the choice of z_i as the earliest vertex in C encountered by the tour. Therefore $T \in \mathrm{TreeA}(M)$, as desired. For later use, define $\mathrm{Tree}(\mu)$ to be the tree T constructed here using the initial letters of the arrival words for μ.

To complete the proof, we show that every object $W = (w^v : v \in V_M) \in S$ has the form $W = \phi_a(\nu)$ for some $\nu \in \mathrm{Par}_M$. Given such a W, note first that we can recover the multigraph M by counting letters in the words w^v. We can also recover the tree $T \in \mathrm{TreeA}(M)$ from W by looking at the initial letters of the words w^v. Next, we execute the algorithm from the end of the last subsection to recover $\mathrm{Bdy}(\nu)$ in reverse. If the algorithm succeeds in consuming all $rn + sn$ edges in the multigraph, then we will have found a ν such that $\phi_a(\nu) = W$. Thus, it suffices to show that the algorithm never gets stuck before using all the edges in M. Suppose the tour being generated by the algorithm has just reached some vertex v (by moving backwards along an edge leading out of v), and suppose that there exists an edge in the multiset E_M that has not yet been consumed by the tour. We consider two cases.

Case 1: $v \neq 0$. Since the tour starts at 0, it follows that the tour has reached v once more than it has left v. Since M is balanced, $\mathrm{indeg}(v) = \mathrm{outdeg}(v)$, and therefore there must be an edge entering v that can be used to continue the reconstruction process.

Case 2: $v = 0$. Note that the partial tour is closed in this case, so it enters each vertex v as often as it leaves v. To get a contradiction, assume that there are no unused edges entering vertex 0. Recall that the edges of T came from the *initial* letters of the arrival words, which are the *last* letters to be consumed by the tour regeneration algorithm. Since there is at least one unused edge by hypothesis, there must exist a nonzero vertex w such that the edge of T leading into w has not yet been consumed.[1] Among all such w, choose one such that $\mathrm{dist}_T(0,w)$ is minimal. Let y be the vertex just before w on the path from 0 to w in T. By choice of y and w, one of the edges from y to w in M was not used by our partial tour P. Since $\mathrm{indeg}(y) = \mathrm{outdeg}(y)$ and P enters y as often as it leaves y, we see that one of the edges leading into y was not used by our partial tour. By definition of the tour regeneration algorithm, it follows that the first arrival edge leading into y (which is an edge of T) was not used. If $y \neq 0$, this contradicts our choice of w. If $y = 0$, this contradicts our assumption that there were no remaining edges entering vertex zero. These contradictions show that there must be an unused edge leading into vertex zero, which can be used to continue the tour reconstruction process. □

Since $\mathrm{Par}_{r,s,n}$ is the disjoint union of the various sets Par_M, we obtain the following fundamental structural result.

Corollary 15. *The arrival map defines a canonical bijection*

$$\mathrm{Par}_{r,s,n} \cong \bigcup_{M \in \mathrm{MGraph}_{r,s,n}} \bigcup_{T \in \mathrm{TreeA}(M)} \prod_{v \in V_M} W_{T_v}(E^{E_{\mathrm{in}}(v,M)} N^{N_{\mathrm{in}}(v,M)}).$$

Dual arguments prove that the departure map is a bijection

$$\phi_d : \mathrm{Par}_M \cong \bigcup_{T \in \mathrm{TreeD}(M)} \prod_{v \in V_M} W(E^{E_{\mathrm{out}}(v,M)} N^{N_{\mathrm{out}}(v,M)})_{T_v}.$$

These maps assemble to give a bijection

$$\mathrm{Par}_{r,s,n} \cong \bigcup_{M \in \mathrm{MGraph}_{r,s,n}} \bigcup_{T \in \mathrm{TreeD}(M)} \prod_{v \in V_M} W(E^{E_{\mathrm{out}}(v,M)} N^{N_{\mathrm{out}}(v,M)})_{T_v}.$$

5 Reformulation of the Partition Statistics

As in the last section, we fix integers r, s, n with $\gcd(r,s) = 1$, and we consider partitions $\mu \in \mathrm{Par}_{r,s,n}$. The next step towards the proof of Theorem 5 is to express the partition statistics $|\mu|$, $\mathrm{mid}_{r/s}(\mu)$, $\mathrm{c}^+_{r/s}(\mu)$, $\mathrm{c}^-_{r/s}(\mu)$, and $\mathrm{ctot}_{r,s}(\mu)$ in terms of the lattice paths, multigraphs, Eulerian tours, arrival words, and departure words from the last section. In this section, we will prove the surprising fact that $|\mu|$, $\mathrm{mid}_{r/s}(\mu)$, and $\mathrm{ctot}_{r/s}(\mu)$ depend only on the multigraph $M(\mu)$, not on the tour $\mathcal{E}(\mu)$. On the other hand, we show that $\mathrm{c}^+_{r/s}(\mu)$ and $\mathrm{c}^-_{r/s}(\mu)$ may be easily calculated using the arrival words and departure words for μ (respectively). Combining these facts with the structure theorems from the last section, we will deduce two fermionic formulas for $F_{r,s,n}$.

To state these results more precisely, we introduce some more definitions. Given a word $w = w_1 w_2 \cdots w_{a+b} \in W(E^a N^b)$, the *inversion set* of w is

$$\mathrm{Inv}(w) = \{(i,j) : 1 \leq i < j \leq a+b \text{ and } w_i = E \text{ and } w_j = N\}.$$

[1]More precisely, this means that the number of times this edge occurs in the partially reconstructed tour is less than the number of times this edge occurs in the edge multiset E_M.

Elements (i,j) of $\mathrm{Inv}(w)$ are called *inversions* of w; we let $\mathrm{inv}(w) = |\mathrm{Inv}(w)|$ be the number of inversions of w. It is well-known that

$$\sum_{w \in W(E^a N^b)} q^{\mathrm{inv}(w)} = \begin{bmatrix} a+b \\ a,b \end{bmatrix}_q = \frac{\prod_{i=1}^{a+b}(1-q^i)}{\prod_{i=1}^{a}(1-q^i)\prod_{i=1}^{b}(1-q^i)}.$$

Next, let $\Lambda(r,s,n)$ denote the unique partition of maximal area in $\mathrm{Par}_{r,s,n}$, and let $A_{\max}(r,s,n) = |\Lambda(r,s,n)|$. It is easy to check that $\Lambda(r,s,n)_i = rn - \lceil ri/s \rceil$ and

$$A_{\max}(r,s,n) = rsn(n-1)/2 + n\sum_{i=0}^{s-1} \lfloor ri/s \rfloor.$$

Given a multigraph $M \in \mathrm{MGraph}_{r,s,n}$, define

$$\begin{aligned}
\mathrm{area}(M) &= A_{\max}(r,s,n) - \sum_{v \in V_M} \lfloor v/s \rfloor N_{\mathrm{out}}(v,M); \\
\mathrm{mid}(M) &= A_{\max}(r,s,n) - \sum_{v,w \in V_M} E_{\mathrm{in}}(v,M) N_{\mathrm{in}}(w,M)\chi(v \geq w); \\
\mathrm{ctot}(M) &= \sum_{v \in V_M} E_{\mathrm{in}}(v,M) N_{\mathrm{in}}(v,M) - (n - E_{\mathrm{in}}(0,M)).
\end{aligned}$$

Theorem 16. *For any $\mu \in \mathrm{Par}_{r,s,n}$, we have:*

$$\mathrm{c}^{+}_{r/s}(\mu) = \sum_{v \in V_M} \mathrm{inv}(w^v(\mu)), \qquad \mathrm{c}^{-}_{r/s}(\mu) = \sum_{v \in V_M} \mathrm{inv}(y^v(\mu));$$

$$|\mu| = \mathrm{area}(M(\mu)), \quad \mathrm{mid}_{r/s}(\mu) = \mathrm{mid}(M(\mu)), \quad \mathrm{ctot}_{r/s}(\mu) = \mathrm{ctot}(M(\mu)).$$

The proof of this theorem is rather long, so we break it up into several subsections. Throughout the proof, we fix $\mu \in \mathrm{Par}_{r,s,n}$ and adopt the following notation. Write $\mathrm{Bdy}(\mu) = w = w_1 w_2 \cdots w_{rn+sn} \in W(E^{rn}N^{sn})$; $\mathrm{VBdy}(\mu) = ((x_0,y_0),\ldots,(x_{rn+sn},y_{rn+sn}))$; $M = M(\mu) = (V_M, E_M)$; and $\mathcal{E} = \mathcal{E}(\mu) = (v_0, v_1, \ldots, v_{rn+sn})$, where $v_i = d_{r,s}(x_i,y_i)$. Let $(e_1, e_2, \ldots, e_{rn+sn})$ be the ordered sequence of edges associated to the Eulerian tour $\mathcal{E}(\mu)$, so that $e_i = (v_{i-1}, v_i)$. Finally, let $(w^v(\mu) : v \in V_M)$ be the sequence of arrival words for μ, and let $(y^v(\mu) : v \in V_M)$ be the sequence of departure words for μ.

5.1 Analysis of c^+ and c^-

Note first that there is a canonical bijection between $\mathrm{dg}(\mu)$ and $\mathrm{Inv}(\mathrm{Bdy}(\mu))$. For, given a cell $c \in \mathrm{dg}(\mu)$, consider the corresponding unit square in $\mathrm{dg}_{r,s,n}(\mu)$. Look for the east step $w_i \in \mathrm{Bdy}(\mu)$ located south of this square and the north step $w_j \in \mathrm{Bdy}(\mu)$ located east of this square. Clearly, (i,j) is an inversion of w that is uniquely determined by the cell in question, and conversely.

The steps w_i and w_j in $\mathrm{Bdy}(\mu)$ correspond to the east edge e_i and the north edge e_j in the Eulerian tour on M. We can compute the quantity $sa(c) - rl(c)$ from the edges e_i and e_j as follows. Recall that w_i is an east step from $(x_i - 1, y_i)$ to (x_i, y_i), while w_j is a north step from $(x_j, y_j - 1)$ to (x_j, y_j). Thus, e_i is an east edge in M from $v_{i-1} = ry_i - sx_i + s$ to $v_i = ry_i - sx_i$, while e_j is a north edge in M from $v_{j-1} = ry_j - r - sx_j$ to $v_j = ry_j - sx_j$. Consideration of $\mathrm{dg}_{r,s,n}(\mu)$ shows that $a(c) = x_j - x_i$ and $l(c) = y_j - 1 - y_i$. Therefore,

$$sa(c) - rl(c) = sx_j - sx_i - ry_j + r + ry_i = v_i - v_{j-1} = v_i - v_j + r = v_{i-1} - v_{j-1} - s.$$

In other words, if $c \in \mathrm{dg}(\mu)$ corresponds to $(i,j) \in \mathrm{Inv}(\mathrm{Bdy}(\mu))$, then we have

$$sa(c) - rl(c) = \mathrm{fin}(e_i) - \mathrm{init}(e_j) = \mathrm{fin}(e_i) - \mathrm{fin}(e_j) + r = \mathrm{init}(e_i) - \mathrm{init}(e_j) - s. \tag{11}$$

Now observe that c contributes to $\mathrm{c}^+_{r/s}(\mu)$ iff $sa(c) - rl(c) = +r$ iff $\mathrm{fin}(e_i) = \mathrm{fin}(e_j)$. Thus, the statistic $\mathrm{c}^+_{r/s}(\mu)$ counts the number of pairs $i < j$ such that e_i is an east edge and e_j is a north edge arriving at the same vertex of M. Such pairs clearly correspond to the inversions in the various arrival words $v^w(\mu)$. This proves the formula for $\mathrm{c}^+_{r/s}(\mu)$ in Theorem 16. Similarly, c contributes to $\mathrm{c}^-_{r/s}(\mu)$ iff $sa(c) - rl(c) = -s$ iff $\mathrm{init}(e_i) = \mathrm{init}(e_j)$. Thus, $\mathrm{c}^-_{r/s}(\mu)$ is the number of pairs $i < j$ such that e_i is an east edge and e_j is a north edge departing from the same vertex of M. The number of such pairs is $\sum_{v \in V_M} \mathrm{inv}(y^v(\mu))$, which proves the second formula in Theorem 16.

At this stage, we can also present a preliminary formula for $\mathrm{mid}_{r/s}(\mu)$. Namely, equation (11) immediately yields

$$\mathrm{mid}_{r/s}(\mu) = \sum_{i<j} \chi(e_i \text{ is an E edge, } e_j \text{ is a N edge, and } -s < \mathrm{fin}(e_i) - \mathrm{init}(e_j) < r).$$

This formula certainly appears to depend on the ordered edge sequence (e_i) in the Eulerian tour, not just on the multigraph M. But we will see shortly that this dependence is illusory.

5.2 Analysis of Area

The next step is to prove that

$$|\mu| = A_{\max}(r,s,n) - \sum_{v \in V_M} \lfloor v/s \rfloor N_{\mathrm{out}}(v,M) = \mathrm{area}(M).$$

Define $\mathrm{area}^c(\mu) = A_{\max}(r,s,n) - |\mu|$, which is the number of lattice squares in the skew shape $\mathrm{dg}_{r,s,n}(\Lambda(r,s,n)) - \mathrm{dg}_{r,s,n}(\mu)$. It suffices to show that $\mathrm{area}^c(\mu) = \sum_{v \in V_M} \lfloor v/s \rfloor N_{\mathrm{out}}(v,M)$. Let us count the number of lattice squares in $\Delta_{r,s,n}$ to the right of a given north step on $\mathrm{Bdy}(\mu)$. Suppose the north step starts at the lattice point (a,b), which maps to the vertex $v = rb - sa$ in the multigraph. The diagonal boundary of $\Delta_{r,s,n}$ has equation $x = (r/s)y$, so that the point $(rb/s, b)$ lies on this boundary. Scanning left from that point to (a,b), it is clear that the number of complete lattice squares to the right of this north step must be

$$\left\lfloor \frac{rb}{s} - a \right\rfloor = \left\lfloor \frac{rb - sa}{s} \right\rfloor = \lfloor v/s \rfloor.$$

Adding over all the north steps in $\mathrm{Bdy}(\mu)$, we obtain the desired expression $\sum_{v \in V_M} \lfloor v/s \rfloor N_{\mathrm{out}}(v,M)$ for the number of cells in the skew shape.

5.3 Analysis of $\Lambda(r,s,n)$

We will prove the last two formulas in Theorem 16 by induction on $\mathrm{area}^c(\mu)$. We handle the base case in this section. Clearly, $\mathrm{area}^c(\mu) = 0$ iff $\mu = \Lambda(r,s,n)$. For convenience, write $\Lambda = \Lambda(r,s,n)$. We first show that

$$\mathrm{mid}_{r/s}(\Lambda) = |\Lambda| = A_{\max}(r,s,n), \qquad \mathrm{c}^+_{r/s}(\Lambda) = \mathrm{c}^-_{r/s}(\Lambda) = \mathrm{ctot}_{r/s}(\Lambda) = 0.$$

It suffices to show that $sa(c) - rl(c) \in (-s, r)$ for every $c \in \mathrm{dg}(\Lambda)$.

By drawing Λ inside $\Delta_{r,s,n}$, we see that $\Lambda_i = rn - \lceil ri/s \rceil$ for $1 \leq i \leq sn$ and $\Lambda'_j = sn - \lceil sj/r \rceil$ for $1 \leq j \leq rn$. Given an arbitrary cell $c = (i,j) \in \mathrm{dg}(\Lambda)$, we compute

$$\begin{aligned} sa(c) - rl(c) &= s(rn - \lceil ri/s \rceil - j) - r(sn - \lceil sj/r \rceil - i) \\ &= (r\lceil sj/r \rceil - sj) + (ri - s\lceil ri/s \rceil). \end{aligned}$$

The first parenthesized quantity lies in the interval $[0,r)$, while the second parenthesized quantity lies in the interval $(-s,0]$, as the reader may readily check using the division algorithm. Thus, $sa(c) - rl(c) \in (-s,r)$, as desired.

We now gather information about the multigraph $M(\Lambda)$ and the Eulerian tour $\mathcal{E}(\Lambda)$. We claim first that every vertex of $M(\Lambda)$ lies in the set $\{0,1,\ldots,r+s-1\}$. If not, the Eulerian tour $\mathcal{E}(\Lambda)$ must take a north edge from some vertex u to a vertex $u + r \geq r + s$. This edge corresponds to a certain north step in $\mathrm{Bdy}(\Lambda)$ starting at a point (x,y) with $d_{r,s}(x,y) = u$. Since $u \geq s$, the point $(x+1,y)$ satisfies $d_{r,s}(x+1,y) \geq 0$ and hence lies in the triangle $\Delta_{r,s,n}$. But then the unit square with southwest corner (x,y) lies inside this triangle and outside Λ, contradicting the definition of Λ.

Let us focus initially on the first $r+s$ steps of Λ, which form a little lattice path P. Let $u_0 = 0, u_1, \ldots, u_{r+s}$ be the vertices in the multigraph visited by the edges of $\mathcal{E}(\Lambda)$ corresponding to the steps of P. Let $\mathcal{E}'$ denote the first $r+s$ edges in the tour $\mathcal{E}(\Lambda)$. We claim that $u_0, \ldots, u_{r+s-1}$ must be pairwise distinct. If not, choose $i < j$ in this range with $u_i = u_j$; note that $0 < j - i < r + s$. Suppose the tour takes a north edges and b east edges to go from u_i to u_j, where $a + b = j - i$. Since $u_j = u_i + ar - bs$ and also $u_i = u_j$, we have $ar = bs > 0$. Since $a + b = j - i < r + s$, we have $a < s$ or $b < r$. Thus, $\mathrm{lcm}(r,s) < rs$ and hence $\gcd(r,s) > 1$, a contradiction. It now follows from the first claim that the list $u_0, \ldots, u_{r+s-1}$ must be a permutation of the vertices $0,1,2,\ldots,r+s-1$. Now, $\mathcal{E}'$ cannot go north from any of the vertices $s, s+1, \ldots, s+r-1$; otherwise $M(\Lambda)$ would have a vertex $\geq r+s$. So

$$\mathcal{E}' \text{ takes east edges from vertices } s, s+1 \ldots, s+r-1 \text{ into vertices } 0,1,\ldots,r-1. \tag{12}$$

Moreover, $\mathcal{E}'$ cannot go east from any of the vertices $0,1,2,\ldots,s-1$; otherwise P would dip below the bounding triangle. So

$$\mathcal{E}' \text{ takes north edges from vertices } 0,1,\ldots,s-1 \text{ into vertices } r, r+1 \ldots, r+s-1. \tag{13}$$

We have now accounted for all the edges of $\mathcal{E}'$. Since there exists an east edge of $\mathcal{E}'$ arriving at vertex 0, and since $u_i \neq u_0 = 0$ for $0 < i < r+s$, we must in fact have $u_{r+s} = 0$. Repeating this argument for the next $r+s$ steps in $\mathrm{Bdy}(\Lambda)$, etc., we see that the full tour $\mathcal{E}(\Lambda)$ just traces out the edge sequence in $\mathcal{E}'$ n times in succession. We conclude that the vertex set of $M(\Lambda)$ is $\{0,1,\ldots,r+s-1\}$ and that the edge multiset of $M(\Lambda)$ is specified by the conditions

$$E_{\mathrm{in}}(u, M(\Lambda)) = n, \quad N_{\mathrm{in}}(u, M(\Lambda)) = 0 \quad \text{for } 0 \leq u < r; \tag{14}$$

$$E_{\mathrm{in}}(u, M(\Lambda)) = 0, \quad N_{\mathrm{in}}(u, M(\Lambda)) = n \quad \text{for } r \leq u < r+s. \tag{15}$$

In particular, there cannot exist vertices $v \geq w$ with $E_{\mathrm{in}}(v, M(\Lambda)) \neq 0$ and $N_{\mathrm{in}}(w, M(\Lambda)) \neq 0$. It is now clear from the definitions that

$$\mathrm{mid}(M(\Lambda)) = A_{\max}(r,s,n) - 0 = \mathrm{mid}_{r/s}(\Lambda);$$

$$\mathrm{ctot}(M(\Lambda)) = 0 - (n-n) = 0 = \mathrm{ctot}_{r/s}(\Lambda).$$

This completes the proof of the base case.

Example 17. The multigraph corresponding to $\Lambda(5,3,3) = (13,11,10,8,6,5,3,1,0)$ is shown in Figure 3.

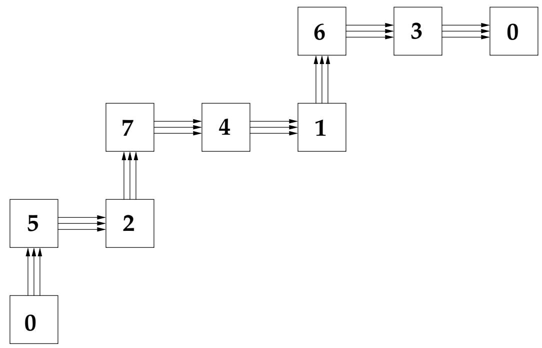

Figure 3: Multigraph for $\Lambda(5,3,3)$.

5.4 Analysis of ctot

In this subsection, we prove that

$$\mathrm{ctot}_{r/s}(\mu) = \sum_{v\in V_M} N_{\mathrm{in}}(v,M)E_{\mathrm{in}}(v,M) - (n - E_{\mathrm{in}}(0,M)) = \mathrm{ctot}(M).$$

We use induction on $\mathrm{area}^c(\mu)$. The base case $\mu = \Lambda(r,s,n)$ has already been proved.

For the induction step, assume $|\mu| < A_{\max}(r,s,n)$. We will apply the induction hypothesis to a certain partition $\mu^* \in \mathrm{Par}_{r,s,n}$ that is obtained from μ by adding one outer corner cell, as follows. The outer corners of μ where we might add a new cell correspond to indices $i < rn + sn$ such that e_i is a north edge and e_{i+1} is an east edge in the Eulerian tour for μ. Addition of the new cell affects the tour by replacing e_i by an east edge and e_{i+1} by a north edge. We therefore need $\mathrm{init}(e_i) \geq s$ so that the new cell will remain inside $\Delta_{r,s,n}$. To define μ^*, consider all the indices i such that e_i and e_{i+1} have the properties just mentioned. (There is at least one such i, since $\mu \neq \Lambda(r,s,n)$.) Among these indices, choose one such that $v_i = \mathrm{fin}(e_i)$ is as large as possible. If there are several choices for i that maximize $\mathrm{fin}(e_i)$, choose i minimal with this property. It is clear that v_i must be the largest vertex in V_M, and e_i is the first north edge of $\mathcal{E}(\mu)$ arriving at this vertex. Accordingly, $E_{\mathrm{in}}(v_i,M) = N_{\mathrm{out}}(v_i,M) = 0$. Now let $\mu^* \in \mathrm{Par}_{r,s,n}$ be the partition whose Eulerian tour is obtained from $\mathcal{E}$ by changing e_i to an east edge and e_{i+1} into a north edge. We write M^* for $M(\mu^*)$, $\mathcal{E}^*$ for $\mathcal{E}(\mu^*)$, etc. The multigraphs M and M^* differ only at vertices v_i, $v_i - r$, $v_i - s$, and $v_i - r - s$. We have $N_{\mathrm{in}}(v_i,M^*) = N_{\mathrm{in}}(v_i,M) - 1$, $E_{\mathrm{out}}(v_i - r, M^*) = E_{\mathrm{out}}(v_i - r, M) + 1$, etc.

By construction, $\mu^* \in \mathrm{Par}_{r,s,n}$ is obtained from μ by the addition of one outer corner cell. By induction hypothesis, we know that $\mathrm{ctot}_{r/s}(\mu^*) = \mathrm{ctot}(M^*)$. Writing $\Delta_1 = \mathrm{ctot}_{r/s}(\mu^*) - \mathrm{ctot}_{r/s}(\mu)$ and $\Delta_2 = \mathrm{ctot}(M^*) - \mathrm{ctot}(M)$, it now suffices to show that $\Delta_1 = \Delta_2$. Let us compute each of these quantities.

Given a vertex $v \in V_M$ and an integer k, let

$$E_{\mathrm{in}}^{<k}(v) = \sum_{j<k} \chi(e_j \text{ is an E edge and } \mathrm{fin}(e_j) = v).$$

Let $E_{\mathrm{in}}^{<k*}(v)$ be the analogous quantity for μ^*, and make analogous definitions for $N_{\mathrm{in}}^{>k}(v)$, etc. Consideration of the arrival words at $v_i - s$ and $v_i - r - s$ shows that

$$\begin{aligned} c^+_{r/s}(\mu^*) - c^+_{r/s}(\mu) &= E^{<i+1*}_{\text{in}}(v_i - s) + N^{>i*}_{\text{in}}(v_i - r - s) - N^{>i+1}_{\text{in}}(v_i - s) \\ &= 0 + N^{>i}_{\text{in}}(v_i - r - s) - N^{>i+1}_{\text{out}}(v_i - r - s). \end{aligned}$$

Similarly, consideration of the departure words at $v_i - r$ and $v_i - r - s$ gives

$$\begin{aligned} c^-_{r/s}(\mu^*) - c^-_{r/s}(\mu) &= N^{>i*}_{\text{out}}(v_i - r) + E^{<i+1*}_{\text{out}}(v_i - r - s) - E^{<i}_{\text{out}}(v_i - r) \\ &= N_{\text{in}}(v_i, M) - 1 + E^{<i+1}_{\text{out}}(v_i - r - s) - E^{<i}_{\text{in}}(v_i - r - s). \end{aligned}$$

Adding, we see that

$$\Delta_1 = N_{\text{in}}(v_i, M) - 1 + N^{>i}_{\text{in}}(v_i - r - s) - N^{>i}_{\text{out}}(v_i - r - s) + E^{\le i}_{\text{out}}(v_i - r - s) - E^{\le i}_{\text{in}}(v_i - r - s).$$

Note that the partial tour consisting of edges $e_1, \ldots, e_i$ enters vertex $v_i - r - s$ as often as it leaves that vertex — unless $v_i - r - s = 0$, in which case there is one more exit than entry. In the current notation, this fact can be written

$$N^{\le i}_{\text{in}}(v_i - r - s) + E^{\le i}_{\text{in}}(v_i - r - s) + \chi(v_i - r - s = 0) = N^{\le i}_{\text{out}}(v_i - r - s) + E^{\le i}_{\text{out}}(v_i - r - s).$$

We can use this relation to rewrite the preceding expression for Δ_1, obtaining

$$N_{\text{in}}(v_i, M) - 1 + N^{>i}_{\text{in}}(v_i - r - s) - N^{>i}_{\text{out}}(v_i - r - s) + N^{\le i}_{\text{in}}(v_i - r - s) - N^{\le i}_{\text{out}}(v_i - r - s) + \chi(v_i - r - s = 0)$$

$$= N_{\text{in}}(v_i, M) - 1 + N_{\text{in}}(v_i - r - s, M) - N_{\text{out}}(v_i - r - s, M) + \chi(v_i - r - s = 0).$$

To compute $\Delta_2 = \text{ctot}(M^*) - \text{ctot}(M)$, first note that

$$-(n - E_{\text{in}}(0, M^*)) - (-(n - E_{\text{in}}(0, M))) = E_{\text{in}}(0, M^*) - E_{\text{in}}(0, M) = \chi(v_i - r - s = 0).$$

Second, note that the only nonzero terms in

$$\sum_{v \in V_M \cup V_{M^*}} [E_{\text{in}}(v, M^*) N_{\text{in}}(v, M^*) - E_{\text{in}}(v, M) N_{\text{in}}(v, M)]$$

come from the vertices $v = v_i - s$ and $v = v_i - r - s$. When $v = v_i - s$, we get the term

$$(E_{\text{in}}(v_i - s, M) - 1)(N_{\text{in}}(v_i - s, M) + 1) - E_{\text{in}}(v_i - s, M) N_{\text{in}}(v_i - s, M)$$

$$= E_{\text{in}}(v_i - s, M) - 1 - N_{\text{in}}(v_i - s, M) = N_{\text{in}}(v_i, M) - 1 - N_{\text{out}}(v_i - r - s, M).$$

When $v = v_i - r - s$, we get the term

$$(E_{\text{in}}(v_i - r - s, M) + 1) N_{\text{in}}(v_i - r - s, M) - E_{\text{in}}(v_i - r - s, M) N_{\text{in}}(v_i - r - s, M) = N_{\text{in}}(v_i - r - s, M).$$

Therefore,

$$\Delta_2 = N_{\text{in}}(v_i, M) - 1 + N_{\text{in}}(v_i - r - s, M) - N_{\text{out}}(v_i - r - s, M) + \chi(v_i - r - s = 0) = \Delta_1.$$

5.5 Analysis of mid

In this subsection, we prove that

$$\mathrm{mid}_{r/s}(\mu) = A_{\max}(r,s,n) - \sum_{v,w \in V_M} E_{\mathrm{in}}(v,M) N_{\mathrm{in}}(w,M) \chi(v \geq w) = \mathrm{mid}(M),$$

which will complete the proof of Theorem 16. We use induction on $\mathrm{area}^c(\mu)$. The base case $\mu = \Lambda(r,s,n)$ has already been proved (§5.3). For the induction step, let μ^* be the partition obtained from μ as in the last subsection. By induction hypothesis, $\mathrm{mid}_{r/s}(\mu^*) = \mathrm{mid}(M^*)$. Writing $\Delta_1 = \mathrm{mid}_{r/s}(\mu^*) - \mathrm{mid}_{r/s}(\mu)$ and $\Delta_2 = \mathrm{mid}(M^*) - \mathrm{mid}(M)$, it suffices to show that $\Delta_1 = \Delta_2$.

Let us begin by computing Δ_2. By maximality of v_i, $v > v_i - s$ implies $E_{\mathrm{in}}(v,M) = E_{\mathrm{out}}(v + s, M) = 0$. So

$$\mathrm{mid}(M) = A_{\max}(r,s,n) + \sum_{v_i - s \geq v \geq w} (-E_{\mathrm{in}}(v,M) N_{\mathrm{in}}(w,M)).$$

A similar formula holds for $\mathrm{mid}(M^*)$. When computing $\Delta_2 = \mathrm{mid}(M^*) - \mathrm{mid}(M)$, we get nonzero contributions from the following summands.

- When $v = w = v_i - s$, the summand for M^* is $-(E_{\mathrm{in}}(v_i - s, M) - 1)(N_{\mathrm{in}}(v_i - s, M) + 1)$ while the summand for M is $-E_{\mathrm{in}}(v_i - s, M) N_{\mathrm{in}}(v_i - s, M)$. Subtracting gives a contribution of $N_{\mathrm{in}}(v_i - s, M) - E_{\mathrm{in}}(v_i - s, M) + 1 = N_{\mathrm{in}}(v_i - s, M) - N_{\mathrm{in}}(v_i, M) + 1$.
- When $v = v_i - s$ and $w < v$, the summand for M^* is $-(E_{\mathrm{in}}(v_i - s, M) - 1) N_{\mathrm{in}}(w,M)$ and the summand for M is $-E_{\mathrm{in}}(v_i - s, M) N_{\mathrm{in}}(w, M)$. Subtracting gives a contribution of $N_{\mathrm{in}}(w,M)$ for each $w < v_i - s$.
- When $v = v_i - r - s$ and $w \leq v$, a similar calculation gives a contribution of $-N_{\mathrm{in}}(w,M)$ for each $w \leq v_i - r - s$.

Adding these contributions and taking cancellation into account, we see that

$$\Delta_2 = 1 - N_{\mathrm{in}}(v_i, M) + \sum_{v_i - r - s < w \leq v_i - s} N_{\mathrm{in}}(w,M). \tag{16}$$

The computation of Δ_1 is a bit more tedious. Recall (§5.1) that

$$\mathrm{mid}_{r/s}(\mu) = \sum_{j<k} \chi(e_j \text{ is an E edge, } e_k \text{ is a N edge, and } -s < \mathrm{fin}(e_j) - \mathrm{init}(e_k) < r).$$

An analogous formula holds for $\mathrm{mid}_{r/s}(\mu^*)$. For most choices of j and k, the summand for μ will equal the corresponding summand for μ^*. The only summands that might not match occur when j or k equals i or $i+1$. Consider the various possible cases.

(A) Let $j = i$ and $k = i + 1$. This pair contributes 0 to $\mathrm{mid}_{r/s}(\mu)$ and 1 to $\mathrm{mid}_{r/s}(\mu^*)$, giving a net contribution of 1 to Δ_1.

(B) Let $j = i$, so that e_j^* is an east edge entering vertex $v = v_i - r - s$ in $\mathcal{E}^*$. Consider the various indices $k > i+1$ such that e_k^* $(= e_k)$ is a north edge. We obtain a certain contribution to $\mathrm{mid}_{r/s}(\mu^*)$ that does not appear for $\mathrm{mid}_{r/s}(\mu)$ since e_j is not an east edge in $\mathcal{E}$. The net contribution to Δ_1 is

$$\sum_w N_{\mathrm{out}}^{>i+1*}(w)\chi(-r < w - (v_i - r - s) < s) = \sum_w N_{\mathrm{out}}^{>i+1}(w)\chi(v_i - 2r - s < w < v_i - r).$$

Replacing w (the initial vertex for the north edge e_k^* in question) by $w + r$ (the final vertex for this edge), we can write this as

$$\sum_w N_{\mathrm{in}}^{>i+1}(w)\chi(v_i - r - s < w < v_i).$$

(C) Let $k = i+1$, so that e_k^* is a north edge leaving vertex $w = v_i - r - s$ in $\mathcal{E}^*$. Consider the various indices $j < i$ such that e_j^* $(= e_j)$ is an east edge. Arguing as above, the net contribution to Δ_1 is

$$\begin{aligned} & \sum_v E_{\mathrm{in}}^{<i*}(v)\chi(-s < v - (v_i - r - s) < r) \\ = \quad & \sum_v E_{\mathrm{in}}^{<i}(v)\chi(v_i - r - 2s < v < v_i - s) \\ = \quad & \sum_v E_{\mathrm{out}}^{<i}(v)\chi(v_i - r - s < v < v_i). \end{aligned}$$

(D) Let $k = i$, so that e_i is a north edge leaving vertex $w = v_i - r$ in $\mathcal{E}$. Consider the various indices $j < i$ such that e_j $(= e_j^*)$ is an east edge. This gives us a contribution to $\mathrm{mid}_{r/s}(\mu)$ but not to $\mathrm{mid}_{r/s}(\mu^*)$. The net contribution to Δ_1 is

$$\begin{aligned} & -\sum_v E_{\mathrm{in}}^{<i}(v)\chi(-s < v - (v_i - r) < r) \\ = \quad & -\sum_v E_{\mathrm{in}}^{<i}(v)\chi(v_i - r - s < v < v_i) \\ = \quad & -\sum_v E_{\mathrm{out}}^{<i}(v)\chi(v_i - r < v < v_i + s) \\ = \quad & -\sum_v E_{\mathrm{out}}^{<i}(v)\chi(v_i - r < v \leq v_i). \end{aligned}$$

The last step follows since no vertex greater than v_i has an east edge leaving it.

(E) Let $j = i+1$, so that e_j is an east edge entering vertex $v = v_i - s$ in $\mathcal{E}$. Consider the various indices $k > i+1$ such that e_k $(= e_k^*)$ is a north edge. As in (D), the net contribution to Δ_1 is

$$\begin{aligned} & -\sum_w N_{\mathrm{out}}^{>i+1}(w)\chi(-r < w - (v_i - s) < s) \\ = \quad & -\sum_w N_{\mathrm{out}}^{>i+1}(w)\chi(v_i - r - s < w < v_i) \\ = \quad & -\sum_w N_{\mathrm{in}}^{>i+1}(w)\chi(v_i - s < w < v_i + r) \\ = \quad & -\sum_w N_{\mathrm{in}}^{>i+1}(w)\chi(v_i - s < w \leq v_i). \end{aligned}$$

Adding the contributions (C) and (D) and noting the cancellation, we get

$$\sum_v E_{\text{out}}^{<i}(v)\chi(v_i - r - s < v \leq v_i - r) - E_{\text{out}}^{<i}(v_i).$$

The subtracted term is zero by minimality of i. Similarly, adding the contributions (B) and (E) and cancelling, we get

$$\sum_w N_{\text{in}}^{>i+1}(w)\chi(v_i - r - s < w \leq v_i - s) - N_{\text{in}}^{>i+1}(v_i).$$

Now $N_{\text{in}}^{>i+1}(v_i) = N_{\text{in}}(v_i, M) - 1$ by minimality of i. Our grand total so far is thus:

$$\Delta_1 = 1 + \sum_v E_{\text{out}}^{<i}(v)\chi(v_i - r - s < v \leq v_i - r) \tag{17}$$

$$+ \sum_w N_{\text{in}}^{>i+1}(w)\chi(v_i - r - s < w \leq v_i - s) + 1 - N_{\text{in}}(v_i, M). \tag{18}$$

Comparing this to the formula (16) for Δ_2, the terms on line (18) look promising, while those on line (17) do not. However, we will show momentarily that

$$1 + \sum_v E_{\text{out}}^{<i}(v)\chi(v_i - r - s < v \leq v_i - r) = \sum_w N_{\text{in}}^{<i}(w)\chi(v_i - r - s < w \leq v_i - s). \tag{19}$$

Using this equality above, together with the fact that $N_{\text{in}}^{=i}(w) = 0 = N_{\text{in}}^{=i+1}(w)$ for w in the indicated range, we discover that

$$\Delta_1 = \sum_w N_{\text{in}}(w, M)\chi(v_i - r - s < w \leq v_i - s) + 1 - N_{\text{in}}(v_i, M) = \Delta_2.$$

Thus we are reduced to verifying (19). Let A be the set of vertices in the multigraph $> v_i - r - s$, and let B be the set of vertices $\leq v_i - r - s$. Consider the first $i - 1$ edges of $\mathcal{E}$. The trail traced out by these edges begins in B (since $v_i \geq r + s$) and ends in A (since e_i starts at vertex $v_i - r$). The edges contributing to the sum $\sum_w N_{\text{in}}^{<i}(w)\chi(v_i - r - s < w \leq v_i - s)$ are precisely the north edges before e_i that go from a vertex in B to a vertex in A. Call these "entering north edges." The edges contributing to the sum $\sum_v E_{\text{out}}^{<i}(v)\chi(v_i - r - s < v \leq v_i - r)$ are precisely the east edges before e_i that go from a vertex in A to a vertex in B. Call these "exiting east edges." As we follow the first $i - 1$ edges, we will alternately encounter entering north edges and exiting east edges (plus other edges that do not concern us). Since this part of the trail ends in A, the last such edge we see must be an entering north edge. Conclusion: There is one more entering north edge than exiting east edge. But this is precisely what (19) is asserting. The proof of Theorem 16 is now complete.

5.6 Fermionic Formulas

Let $F_M(q, z, w, y) = \sum_{\mu \in \text{Par}_M} q^{|\mu|} z^{\text{mid}_{r/s}(\mu)} w^{\text{c}^+_{r/s}(\mu)} y^{\text{c}^-_{r/s}(\mu)}$. By combining Theorem 14 and Theorem 16, we obtain the identity

$$F_M = q^{\text{area}(M)} z^{\text{mid}(M)} y^{\text{ctot}(M)} \sum_{T \in \text{TreeA}(M)} \prod_{v \in V_M} \begin{bmatrix} E_{\text{in}}(v, M) + N_{\text{in}}(v, M) - 1 \\ E_{\text{in}}(v, M)', N_{\text{in}}(v, M)' \end{bmatrix}_{w/y} (w/y)^{\text{pow}},$$

where $E_{\text{in}}(v,M)' = E_{\text{in}}(v,M) - \chi(T_v = E)$, $N_{\text{in}}(v,M)' = N_{\text{in}}(v,M) - \chi(T_v = N)$, and $\text{pow} = N_{\text{in}}(v,M)\chi(T_v = E)$. Similarly, we have the dual identity

$$F_M = q^{\text{area}(M)} z^{\text{mid}(M)} w^{\text{ctot}(M)} \sum_{T \in \text{TreeD}(M)} \prod_{v \in V_M} \begin{bmatrix} E_{\text{in}}(v,M) + N_{\text{in}}(v,M) - 1 \\ E_{\text{in}}(v,M)', N_{\text{in}}(v,M)' \end{bmatrix}_{y/w} (y/w)^{\text{pow}'},$$

where $\text{pow}' = E_{\text{in}}(v,M)\chi(T_v = N)$. Adding over all $M \in \text{MGraph}_{r,s,n}$, we deduce two fermionic formulas for $F_{r,s,n}(q,z,w,y)$:

Theorem 18.

$$\begin{aligned} F_{r,s,n} &= \sum_{M \in \text{MGraph}_{r,s,n}} q^{\text{area}(M)} z^{\text{mid}(M)} y^{\text{ctot}(M)} \sum_{T \in \text{TreeA}(M)} \prod_{v \in V_M} \begin{bmatrix} E_{\text{in}}(v,M) + N_{\text{in}}(v,M) - 1 \\ E_{\text{in}}(v,M)', N_{\text{in}}(v,M)' \end{bmatrix}_{w/y} (w/y)^{\text{pow}} \\ &= \sum_{M \in \text{MGraph}_{r,s,n}} q^{\text{area}(M)} z^{\text{mid}(M)} w^{\text{ctot}(M)} \sum_{T \in \text{TreeD}(M)} \prod_{v \in V_M} \begin{bmatrix} E_{\text{in}}(v,M) + N_{\text{in}}(v,M) - 1 \\ E_{\text{in}}(v,M)', N_{\text{in}}(v,M)' \end{bmatrix}_{y/w} (y/w)^{\text{pow}'}. \end{aligned}$$

6 Proof of Theorem 5

In this section, we will prove the crucial symmetry property $F_{r,s,n}(q,z,w,y) = F_{r,s,n}(q,z,y,w)$ by constructing an involution I on $\text{Par}_{r,s,n}$ that fixes area and $\text{mid}_{r/s}$ while interchanging $\text{c}^+_{r/s}$ and $\text{c}^-_{r/s}$. Since area, $\text{mid}_{r/s}$, and $\text{ctot}_{r/s}$ are constant on the subsets Par_M (for $M \in \text{MGraph}_{r,s,n}$), it suffices to construct involutions $I_M : \text{Par}_M \to \text{Par}_M$ such that $\text{c}^+_{r/s}(I_M(\mu)) = \text{ctot}(M) - \text{c}^+_{r/s}(\mu)$ for all $\mu \in \text{Par}_M$.

6.1 Definition of the Involution

Fix $M \in \text{MGraph}_{r,s,n}$ and $\mu \in \text{Par}_M$. Let $T = \text{Tree}(\mu)$ be the oriented tree leading from 0 constructed from the initial letters of the arrival words $w^v(\mu)$ in §4.5. Recall that $T_v = w^v(\mu)_1$ gives the direction (N or E) of the first arrival edge leading into vertex v. We now use T to separate the nonzero vertices of M into three disjoint classes.

1. Call a nonzero vertex v *red* iff $v + s \notin V_M$ or $v + s \in V_M$ and $\text{dist}_T(0, v+s) \neq \text{dist}_T(0,v) - 1$.
2. Call a nonzero vertex v *blue* iff $v - r \notin V_M$ or $v - r \in V_M$ and $\text{dist}_T(0, v-r) \neq \text{dist}_T(0,v) - 1$.
3. Call a nonzero vertex v *green* iff neither of the previous conditions holds. This means that $v + s$ and $v - r$ are vertices of M, and $\text{dist}_T(0, v+s) = \text{dist}_T(0, v-r) = \text{dist}_T(0,v) - 1$.

A convenient way to visualize this situation is to embed T in $\mathbb{R}^2$ by placing vertex 0 at $(0,0)$ and then drawing the unique paths (consisting of north and east steps) leading to all the other vertices. Each vertex $v \in V_M$ will appear in this picture at some point (x,y) in the plane with $d_{r,s}(x,y) = v$. One can check that a nonzero vertex v located at (x,y) is red iff $(x-1,y)$ is not in the tree; v is blue iff $(x,y-1)$ is not in the tree; and v is green iff both $(x-1,y)$ and $(x,y-1)$ are vertices of this tree.

Now let v be a green vertex. Then the unique path in T from 0 to v goes through either $v+s$ or $v-r$ just before reaching v. Suppose we modify T by replacing T_v by the opposite letter. It is easy to check that the result is another oriented tree T' such that all vertices are assigned the same color as before. More generally, if we modify T by simultaneously toggling the edges T_v at an arbitrary subset of the green vertices, the result is another tree $T' \in \text{TreeA}(M)$ with the same color assignment as before.

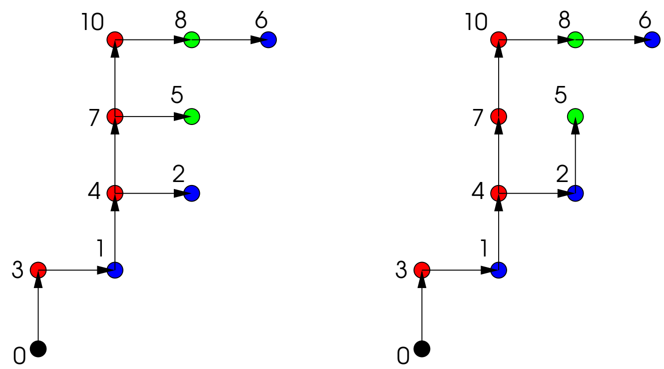

Figure 4: Trees associated to μ and $I(\mu)$.

Now we are ready to define the map I_M. First, define *reversal maps* $\mathrm{rev} : W(E^aN^b) \to W(E^aN^b)$ and $\mathrm{rev}' : W(E^aN^b) \to W(E^aN^b)$ by setting

$$\mathrm{rev}(w_1w_2\dots w_{a+b}) = w_{a+b}\cdots w_2w_1, \qquad \mathrm{rev}'(w_1w_2w_3\dots w_{a+b}) = w_1w_{a+b}\cdots w_3w_2.$$

Note that rev reverses an entire binary word, while rev' reverses the letters in a word following the initial letter. Obviously, rev is an involution on $W(E^aN^b)$, while rev' is an involution on the sets $W_E(E^aN^b)$ and $W_N(E^aN^b)$. To compute $I_M(\mu)$, first find T and the vertex colorings as above. Replace $w^v(\mu)$ by $\mathrm{rev}(w^v(\mu))$ at every green vertex v, and replace $w^v(\mu)$ by $\mathrm{rev}'(w^v(\mu))$ at every other vertex v of V_M. The initial letters of the new arrival words determine a new tree T', as argued above. Therefore, Theorem 14 guarantees that there is a unique partition $I_M(\mu) \in \mathrm{Par}_M$ associated to the new arrival words. Moreover, since the coloring of the vertices relative to T' is the same as the coloring relative to T, it is immediate that I_M is an involution. Define I to be the involution on $\mathrm{Par}_{r,s,n}$ obtained by assembling the various maps I_M.

Example 19. Let $\mu = (12, 12, 10, 8, 7, 4, 1, 1, 1) \in \mathrm{Par}_{3,2,5}$. To compute $I(\mu)$, we first draw $\mathrm{Bdy}(\mu)$ (see Figure 1) and the multigraph $M(\mu)$ (see Figure 2). As in Example 10, we find that the arrival words for μ are:

$$w^0 = EE,\ w^1 = EEE,\ w^2 = EEE,\ w^3 = NEEN,\ w^4 = NENNE,$$
$$w^5 = EN,\ w^6 = EN,\ w^7 = NN,\ w^8 = E,\ w^{10} = N.$$

Looking at the intial letters of these arrival words, we draw the tree $T = \mathrm{Tree}(\mu)$ in $\mathbb{R}^2$ as shown on the left in Figure 4. The red vertices are 3, 4, 7, and 10; the blue vertices are 1, 2, and 6; the green vertices are 5 and 8. We therefore fully reverse w^5 and w^8, and reverse all but the first letter of the remaining words. The new arrival words are

$$w^0 = EE,\ w^1 = EEE,\ w^2 = EEE,\ w^3 = NNEE,\ w^4 = NENNE,$$

$$w^5 = NE,\ w^6 = EN,\ w^7 = NN,\ w^8 = E,\ w^{10} = N.$$

The associated tree appears on the right in Figure 4. Decoding these arrival words as in Example 12, we find that $I(\mu) = (12, 10, 10, 8, 7, 6, 1, 1, 1)$.

6.2 Analysis of $\mathrm{c}^+_{r/s}$

To finish the proof of Theorem 5, we need only verify that $\mathrm{c}^+_{r/s}(I_M(\mu)) = \mathrm{ctot}(M) - \mathrm{c}^+_{r/s}(\mu)$ for all $M \in \mathrm{MGraph}_{r,s,n}$ and all $\mu \in \mathrm{Par}_M$. Fix M and μ, and let $T = \mathrm{Tree}(\mu)$. Recall that $\mathrm{c}^+_{r/s}(\mu) = \sum_{v\in V_M} \mathrm{inv}(w^v(\mu))$, and similarly for $\mathrm{c}^+_{r/s}(I_M(\mu))$. Now, it is easy to check that

$$\begin{array}{lll} \mathrm{inv}(\mathrm{rev}(w)) & = ab - \mathrm{inv}(w) & \text{for all } w \in W(E^aN^b); \\ \mathrm{inv}(\mathrm{rev}'(w)) & = ab - \mathrm{inv}(w) + b & \text{for all } w \in W_E(E^aN^b); \\ \mathrm{inv}(\mathrm{rev}'(w)) & = ab - \mathrm{inv}(w) - a & \text{for all } w \in W_N(E^aN^b). \end{array}$$

Furthermore, it is clear that $T_v = w^v(\mu)_1 = N$ if v is a red vertex, while $T_v = w^v(\mu)_1 = E$ if v is a blue vertex or $v = 0$. From these remarks and the definition of I_M, it follows that

$$\mathrm{c}^+_{r/s}(I_M(\mu)) = \sum_{v\in V_M} E_{\mathrm{in}}(v,M)N_{\mathrm{in}}(v,M) + \sum_{\text{blue } v} N_{\mathrm{in}}(v,M) - \sum_{\text{red } v} E_{\mathrm{in}}(v,M) - \mathrm{c}^+_{r/s}(\mu).$$

On the other hand,

$$\mathrm{ctot}(M) = \sum_{v\in V_M} E_{\mathrm{in}}(v,M)N_{\mathrm{in}}(v,M) - (n - E_{\mathrm{in}}(0,M)).$$

Comparing these expressions, we see that everything reduces to the following lemma.

Lemma 20.

$$E_{\mathrm{in}}(0,M) + \sum_{\text{red } v} E_{\mathrm{in}}(v,M) - \sum_{\text{blue } v} N_{\mathrm{in}}(v,M) = n.$$

Proof. Let $(v_0 = 0, v_1, \ldots, v_{rn+sn} = 0)$ be the sequence of vertices in the Eulerian tour for μ, and let $(e_1, \ldots, e_{rn+sn})$ be the sequence of edges in this tour. Define $d_i = \mathrm{dist}_T(0, v_i)$ for $0 \le i \le (r+s)n$. We make three claims about these distances.

(A) If e_i belongs to the edge set of T or if e_i enters a green vertex of T, then $d_i = d_{i-1} + 1$.

(B) If e_i is an east edge entering vertex 0 or a red vertex of T, then $d_i = d_{i-1} + 1 - (r+s)$.

(C) If e_i is a north edge entering a blue vertex of T, then $d_i = d_{i-1} + 1 + (r+s)$.

Claim (A) is clear; the other two claims will be proved in a moment. Denote the number of edges in the tour satisfying the hypotheses of (A), (B), and (C) by n_0, n_1, and n_2, respectively. Clearly, $n_1 = E_{\mathrm{in}}(0,M) + \sum_{\text{red } v} E_{\mathrm{in}}(v,M)$ and $n_2 = \sum_{\text{blue } v} N_{\mathrm{in}}(v,M)$. Thus we must prove that $n_1 - n_2 = n$. Every edge e_i belongs to exactly one of the categories (A), (B), or (C), and hence $n_0 + n_1 + n_2 = (r+s)n$. Furthermore, $0 = d_{rn+sn} = \sum_{i=1}^{rn+sn}(d_i - d_{i-1})$. Adding up the contributions from the three types of edges, we get

$$n_0 + n_1(1-r-s) + n_2(1+r+s) = 0.$$

It follows that $(r+s)(n_1 - n_2) = n_0 + n_1 + n_2 = (r+s)n$, and hence $n_1 - n_2 = n$.

Claims (B) and (C) follow from a topological argument illustrated in Figure 5 in the case $(r,s) = (4,3)$. We draw the vertices of $M(\mu)$ and the edges of $T = \mathrm{Tree}(\mu)$ between the lines $x + y = 0$ and $x + y = r + s$, as explained in §4.3. We view this region as a cylinder obtained by identifying each point (a,b) on the line $x + y = r + s$ with the point $(a - r, b - s)$ on the line $x + y = 0$. For each vertex $v \in V_M$, there is a unique path in T from 0 to v. We define the *winding number of v relative to T* to be the number of times this path "wraps around" the cylinder by jumping from the line $x + y = r + s$ back to the line $x + y = 0$. Denote this number by $\mathrm{wind}(v, T)$. We allow a nonzero vertex v that is divisible by $r + s$ to have two winding numbers: namely, the copy of v on the line $x + y = 0$ has winding number one greater than the copy of v on the line $x + y = r + s$. See Figure 5. We now make the following observations. (In the following discussion, if $r + s$ divides v_{i-1} or v_i, the location of e_i in the picture determines, in the obvious way, which winding numbers to use.)

(i) Suppose $w \in V_M$ and there are two trails from 0 to w in M of lengths m_1 and m_2. Then $r + s$ divides $m_1 - m_2$. For suppose the first trail takes a_1 north edges and b_1 east edges, while the second trail takes a_2 north edges and b_2 east edges. Then $a_1 r - b_1 s = w = a_2 r - b_2 s$, so that $(a_1 - a_2)r = (b_1 - b_2)s$. Since $\mathrm{lcm}(r, s) = rs$, it follows that $a_1 - a_2 = ks$ and $b_1 - b_2 = kr$ for some integer $k \geq 0$. So $m_1 - m_2 = (a_1 + b_1) - (a_2 + b_2) = k(r + s)$.

(ii) Suppose $v \in V_M$, and write $\mathrm{dist}_T(0, v) = q(r + s) + u$ where $0 \leq u < r + s$. If v lies below the line $x+y = r+s$, then $\mathrm{wind}(v, T) = q$. If v lies on the line $x+y = r+s$, then $\mathrm{wind}(v, T) = q-1$. This follows from the definition of winding number and the fact that it always takes exactly $r + s$ steps to go from the line $x + y = 0$ to the line $x + y = r + s$.

(iii) For each i, $\mathrm{wind}(v_i, T) - \mathrm{wind}(v_{i-1}, T) \in \{-1, 0, 1\}$. For, it is geometrically evident from the picture of T on the cylinder that there is no way for the winding number to change by two or more when following a single edge of M.

(iv) If $\mathrm{wind}(v_i, T) = \mathrm{wind}(v_{i-1}, T)$ and e_i is not in T, then v_i is a green vertex. For it follows easily from (ii) that $\mathrm{dist}_T(0, v_{i-1}) = \mathrm{dist}_T(0, v_i) - 1$ in this situation.

(v) Suppose e_i is not in the edge set of T, and v_i is either zero or a red vertex. Then e_i is an east edge, and it readily follows from (iii) and (iv) that $\mathrm{wind}(v_i, T) = \mathrm{wind}(v_{i-1}, T) - 1$. Consider the following two directed paths in M from 0 to v_i. The first path is the unique path in T from 0 to v_i, of length d_i. The second path is the path in T from 0 to v_{i-1}, followed by the edge e_i; the length of this path is $d_{i-1} + 1$. Using (i) and (ii), we easily deduce that $d_i = d_{i-1} + 1 - (r + s)$. Thus claim (B) holds.

(vi) Suppose e_i is not in the edge set of T, and v_i is a blue vertex. Then e_i is a north edge, and it readily follows from (iii) and (iv) that $\mathrm{wind}(v_i, T) = \mathrm{wind}(v_{i-1}, T) + 1$. Consider the following two directed paths in M from 0 to v_i. The first path is the unique path in T from 0 to v_i, of length d_i. The second path is the path in T from 0 to v_{i-1}, followed by the edge e_i; the length of this path is $d_{i-1} + 1$. Using (i) and (ii), we easily deduce that $d_i = d_{i-1} + 1 + (r + s)$. Thus claim (C) holds.

□

6.3 The Combinatorial Homotopy

For each critical rational r/s, we now have an involution $I = I_{r/s}$ that switches $\mathrm{h}^+_{r/s}$ and $\mathrm{h}^-_{r/s}$ while preserving area. By composing these involutions, we can produce bijections proving the equidistribution of any two statistics h^δ_x and $h^{\delta'}_{x'}$. For example, Figure 6 shows how these involutions act on a particular object as the parameter value x goes from 0 to ∞.

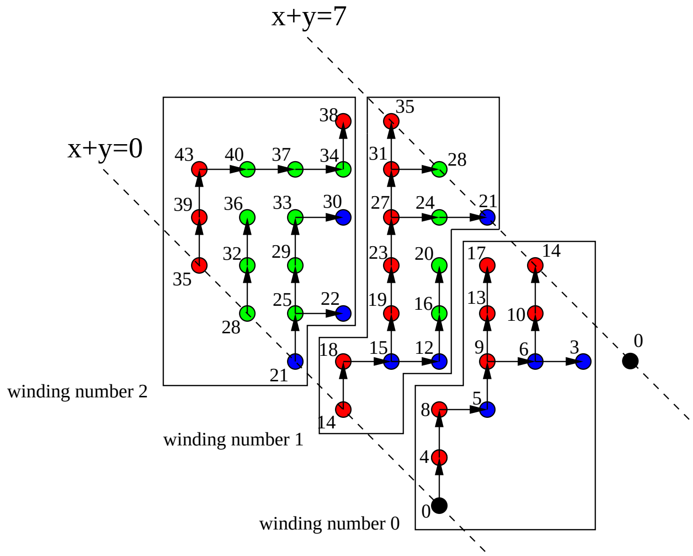

Figure 5: Proof of claims (B) and (C).

7 Connection to q, t-Catalan Numbers

Definition 21. Let r, s, n be positive integers with $\gcd(r, s) = 1$. The *rational-slope* q, t*-Catalan numbers* are the polynomials

$$C_{r,s,n}(q,t) = \sum_{\mu \in \mathrm{Par}_{r,s,n}} q^{\mathrm{area}^c(\mu)} t^{\mathrm{h}^+_{r/s}(\mu)}.$$

When r/s is an integer (i.e., $s = 1$), one can show that this definition agrees with the combinatorial interpretation of the q, t-Catalan number $C_n^{(r)}(q, t)$ first proposed by Mark Haiman. More specifically, if P is an $r/1$-Dyck path of order n, then the statistics $\mathrm{area}(P)$ and $\mathrm{dinv}_r(P)$ defined in [12] are respectively equal to $\mathrm{area}^c(\mu)$ and $\mathrm{h}^+_r(\mu)$, where $\mathrm{Bdy}(\mu) = P$ (cf. Lemma 6.3.3 in [7]). We now present an extension of a fundamental conjecture about the combinatorial q, t-Catalan numbers.

Conjecture 22. *For all* r, s, n *as above, we have the* joint symmetry property

$$C_{r,s,n}(q,t) = C_{r,s,n}(t,q).$$

At present, this conjecture has only been proved for $r = s = 1$. More specifically, Garsia and Haglund proved that $C_{1,1,n}(q, t)$ is the Hilbert series for the doubly graded S_n-module of diagonal harmonic alternants [4, 5]. That Hilbert series is manifestly symmetric in q and t, whence the result. Even when $r = s = 1$, it is an open problem to construct an explicit bijection on $\mathrm{Par}_{r,s,n}$ that interchanges area^c and $\mathrm{h}^+_{r/s}$. On the other hand, for all r and n, there are known bijections on $\mathrm{Par}_{r,1,n}$ that send area^c to $\mathrm{h}^+_{r/s}$ or vice versa [12]. These maps prove the *univariate* symmetry $C_{r,1,n}(q, 1) = C_{r,1,n}(1, q)$.

There is a remarkable connection between the symmetry conjecture given here and the equidistribution property in Theorem 4. More precisely, we now show that certain cases of the theorem follow easily from corresponding cases of the conjecture.

Theorem 23. *Fix a positive integer* $r \geq 1$. *If* $C_{r,1,n}(q, t) = C_{r,1,n}(t, q)$ *for all sufficiently large* n*, then the statistics* $\ell(\mu)$ *and* $\mathrm{h}^+_r(\mu)$ *are equidistributed on* $\mathrm{Par}(k)$ *for all* $k \geq 0$.

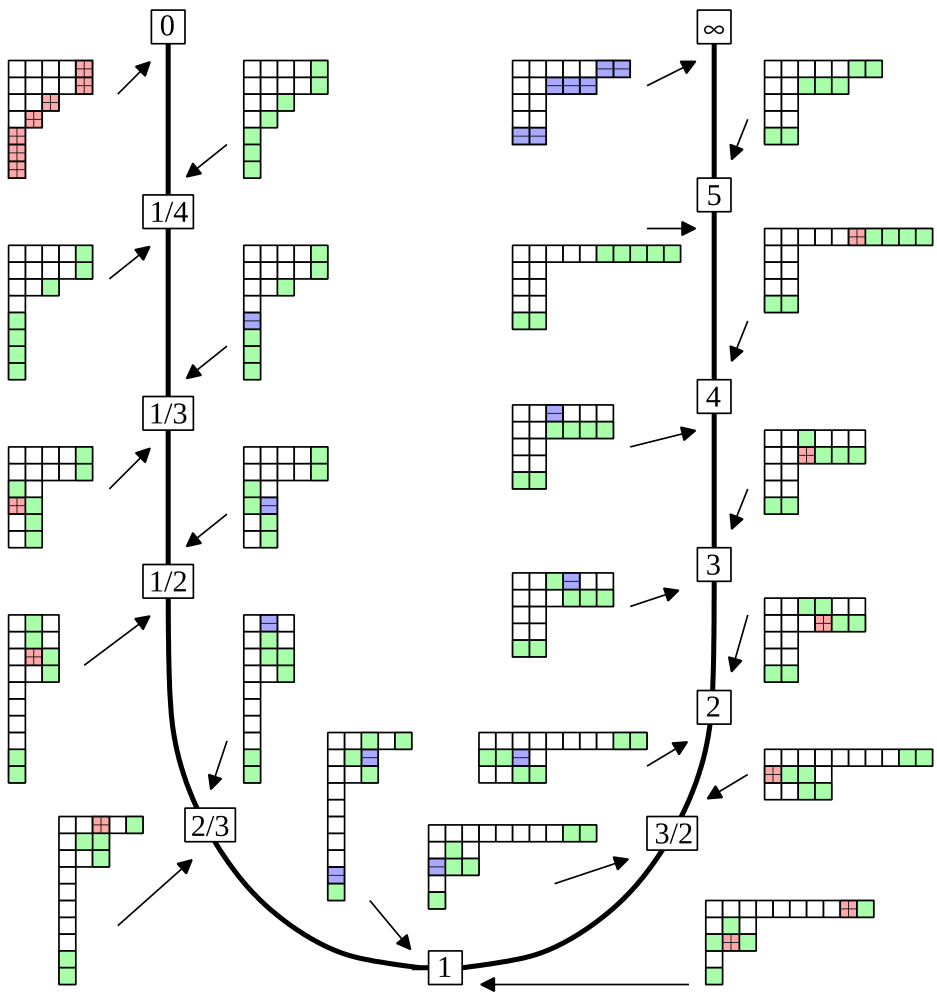

Figure 6: Example of the combinatorial homotopy.

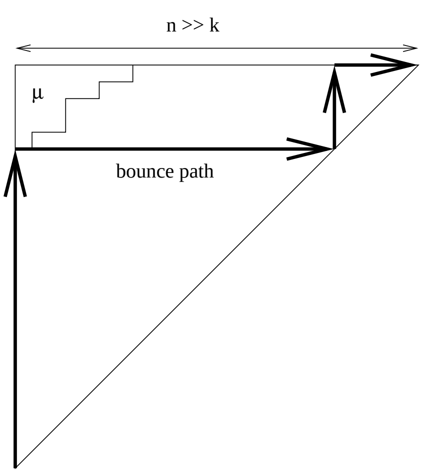

Figure 7: Bouncing through a large triangle.

Proof. For simplicity, we only consider the case $r = 1$. Write $C_n(q,t)$ for $C_{1,1,n}(q,t)$. The proof uses an alternate combinatorial interpretation of the q,t-Catalan number due originally to Haglund [6]. Given $\mu \in \mathrm{Par}_{1,1,n}$, define $\mathrm{bounce}(\mu)$ by the following construction. Draw the diagram $\mathrm{dg}_{1,1,n}(\mu)$ in the triangle $\Delta_{1,1,n}$, as usual. A ball starts at $(0,0)$ and moves north k_0 units until it touches either the southwest corner of a unit square in $\mathrm{dg}_{1,1,n}(\mu)$ or the top boundary line $y = n$. The ball then bounces east k_0 units to (k_0, k_0). If $k_0 < n$, the ball repeats this process, moving north k_1 units until it touches the southwest corner of a square of μ or the line $y = n$, and then moving east k_1 units to $(k_0 + k_1, k_0 + k_1)$. This bouncing process continues, generating a sequence $(k_0, k_1, \ldots, k_s)$, until the ball finally reaches (n, n). Haglund's *bounce statistic* is given by either of the formulas

$$\mathrm{bounce}(\mu) = \sum_{i=0}^{s} i k_i = \sum_{i=0}^{s} (n - k_i).$$

(Note that the bounce statistic depends both on μ and on n.)

The hypothesis $C_n(q,t) = C_n(t,q)$ implies that there is a bijection on $\mathrm{Par}_{1,1,n}$ that interchanges area^c and h_1^+. On the other hand, it is known [12] that there is a bijection on $\mathrm{Par}_{1,1,n}$ such that $(\mathrm{area}^c, \mathrm{h}_1^+)$ maps to $(\mathrm{bounce}, \mathrm{area}^c)$. Composing these bijections, we get a bijection $\alpha : \mathrm{Par}_{1,1,n} \to \mathrm{Par}_{1,1,n}$ such that $(\mathrm{area}^c, \mathrm{h}_1^+)$ maps to $(\mathrm{area}^c, \mathrm{bounce})$.

Fix $k \geq 0$, and suppose $n \geq 2k$. On one hand, this choice of n ensures that $\mathrm{Par}(k) \subseteq \mathrm{Par}_{1,1,n}$. On the other hand, this choice of n guarantees that $\mathrm{bounce}(\mu) = \ell(\mu)$ for $\mu \in \mathrm{Par}(k)$, because the bouncing ball will reach (n,n) after only two bounces. See Figure 7 for an example. It follows that α restricts to a bijection on $\mathrm{Par}(k)$ that sends $\mathrm{h}_1^+(\mu)$ to $\ell(\mu) = \mathrm{h}_0^+(\mu)$. Thus we have a new bijective proof of the equidistribution of h_1^+ and h_0^+ (depending, of course, on the unknown bijection interchanging area^c and h_1^+!).

When $r > 1$, an analogous proof can be given using the "higher-order" bounce statistics introduced in [12]. The key point is that these bounce statistics also reduce to $\ell(\mu)$ when n is sufficiently large compared to $|\mu|$. □

Acknowledgements

The authors express their deep gratitude to Mark Haiman for first bringing their attention to Theorem 3 in 2001, and for his continuing encouragement and support. We also thank Jeff Remmel and Jim Haglund for many helpful discussions regarding q,t-Catalan numbers and related topics.

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