Rational q-Catalan numbers and q-binomials

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$$\begin{bmatrix}
. & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \\
. & . & . & . & . & . & . & . & . & . & 1 & 1 & 1 & 1 & . \\
. & . & . & . & . & . & . & . & 2 & 2 & 2 & 1 & 1 & . & . \\
. & . & . & . & . & . & 1 & 3 & 3 & 2 & 1 & 1 & . & . & . \\
. & . & . & . & . & 2 & 4 & 3 & 2 & 1 & 1 & . & . & . & . \\
. & . & . & . & 2 & 4 & 3 & 2 & 1 & 1 & . & . & . & . & . \\
. & . & . & 1 & 4 & 3 & 2 & 1 & 1 & . & . & . & . & . & . \\
. & . & . & 3 & 3 & 2 & 1 & 1 & . & . & . & . & . & . & . \\
. & . & 2 & 3 & 2 & 1 & 1 & . & . & . & . & . & . & . & . \\
. & . & 2 & 2 & 1 & 1 & . & . & . & . & . & . & . & . & . \\
. & 1 & 2 & 1 & 1 & . & . & . & . & . & . & . & . & . & . \\
. & 1 & 1 & 1 & . & . & . & . & . & . & . & . & . & . & . \\
. & 1 & 1 & . & . & . & . & . & . & . & . & . & . & . & . \\
. & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\
1 & . & . & . & . & . & . & . & . & . & . & . & . & . & .
\end{bmatrix}$$

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Goal

Find combinatorial interpretations for rational q-Catalan numbers and q-binomials.

q-binomials

$$[n]_q = 1 + q + q^2 + \cdots + q^{n-1},$$

$$[n]!_q = [n]_q[n-1]_q \cdots [2]_q[1]_q,$$

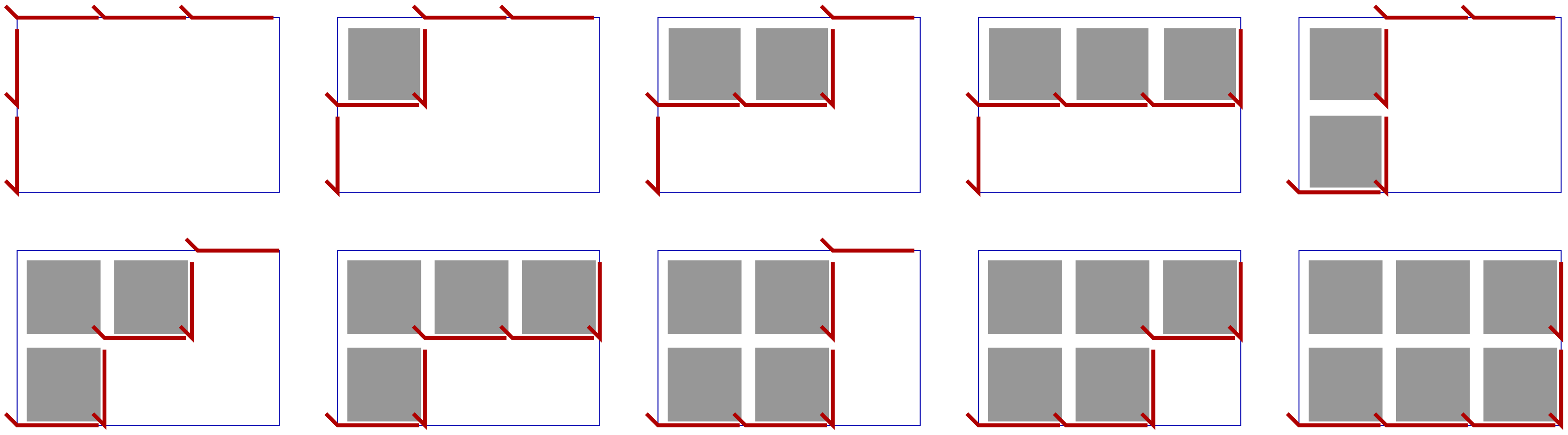
$$\begin{bmatrix} a+b \\ a, b \end{bmatrix}_q = \frac{[a+b]!_q}{[a]!_q[b]!_q}.$$

q-binomials

Theorem:

$$\begin{bmatrix} a+b \\ a, b \end{bmatrix}_q = \sum_{\mu \in \mathcal{R}^{\mathrm{ptn}}(a,b)} q^{|\mu|}.$$

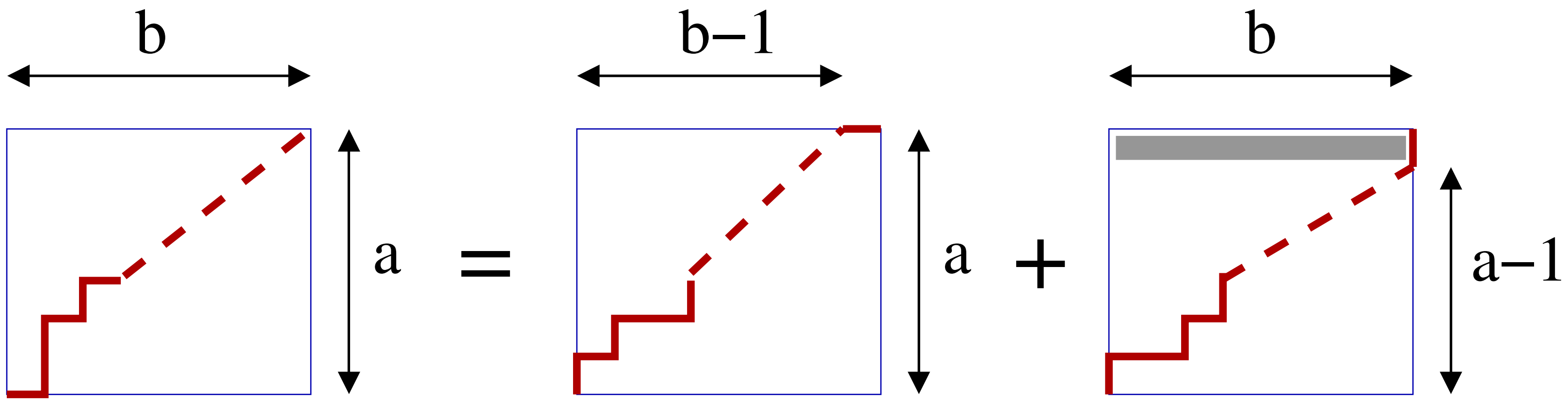
Example: $a = 2, b = 3$



$$\begin{bmatrix} 2+3 \\ 2,3 \end{bmatrix}_q = 1 + q + 2q^2 + 2q^3 + 2q^4 + q^5 + q^6.$$

Proof

$$\begin{bmatrix} a+b \\ a,b \end{bmatrix}_q = \begin{bmatrix} a+b-1 \\ a,b-1 \end{bmatrix}_q + q^b \begin{bmatrix} a+b-1 \\ a-1,b \end{bmatrix}_q .$$

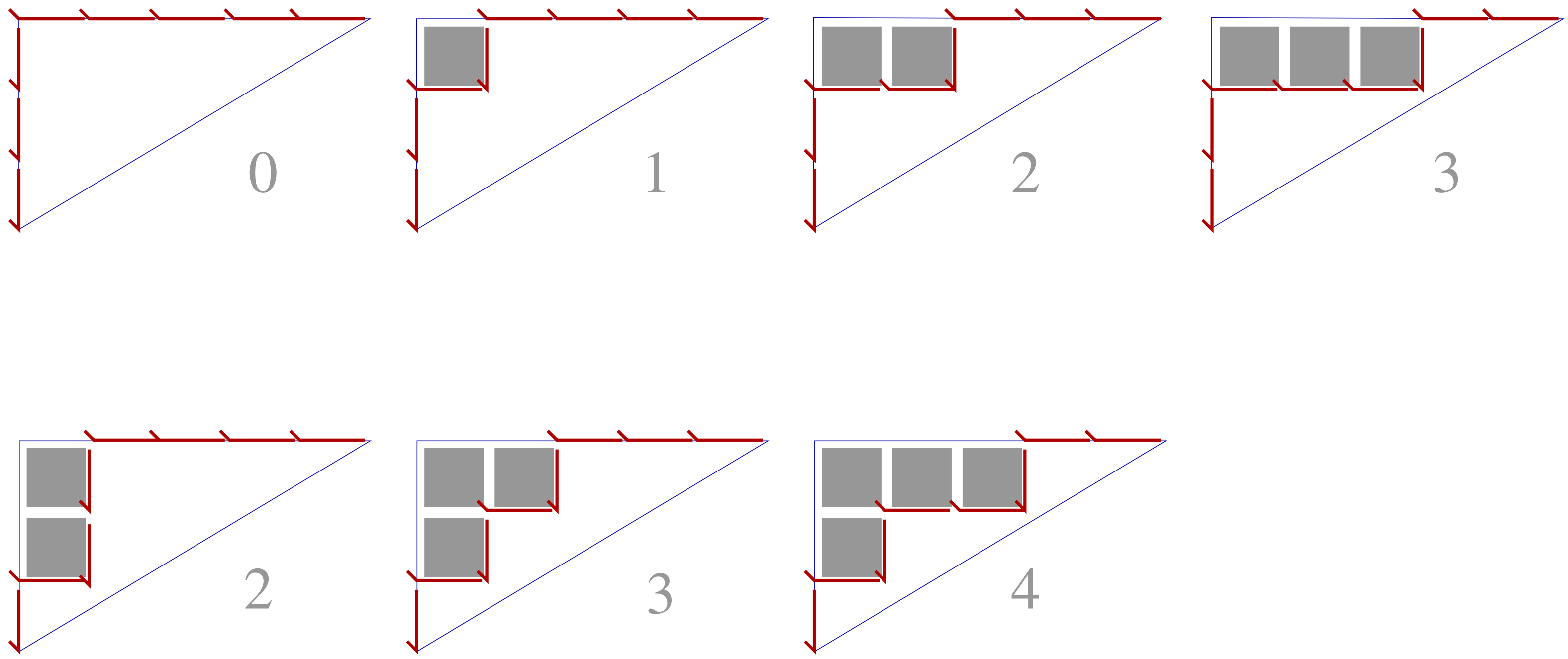


q, t-Catalan

$$\mathsf{Cat}_{a,b}(q) = \frac{1}{[a+b]_q} \begin{bmatrix} a+b \\ a, b \end{bmatrix}_q.$$

$$\begin{aligned} \mathsf{Cat}_{3,5}(q) &= \frac{1}{[8]_q} \frac{[8]_q [7]_q [6]_q}{[3]_q [2]_q [1]_q} \\ &= \frac{1-q^7}{1-q} \frac{1-q^6}{1-q} \frac{1-q}{1-q^3} \frac{1-q}{1-q^2} \\ &= 1 + q^2 + q^3 + q^4 + q^5 + q^6 + q^8. \end{aligned}$$

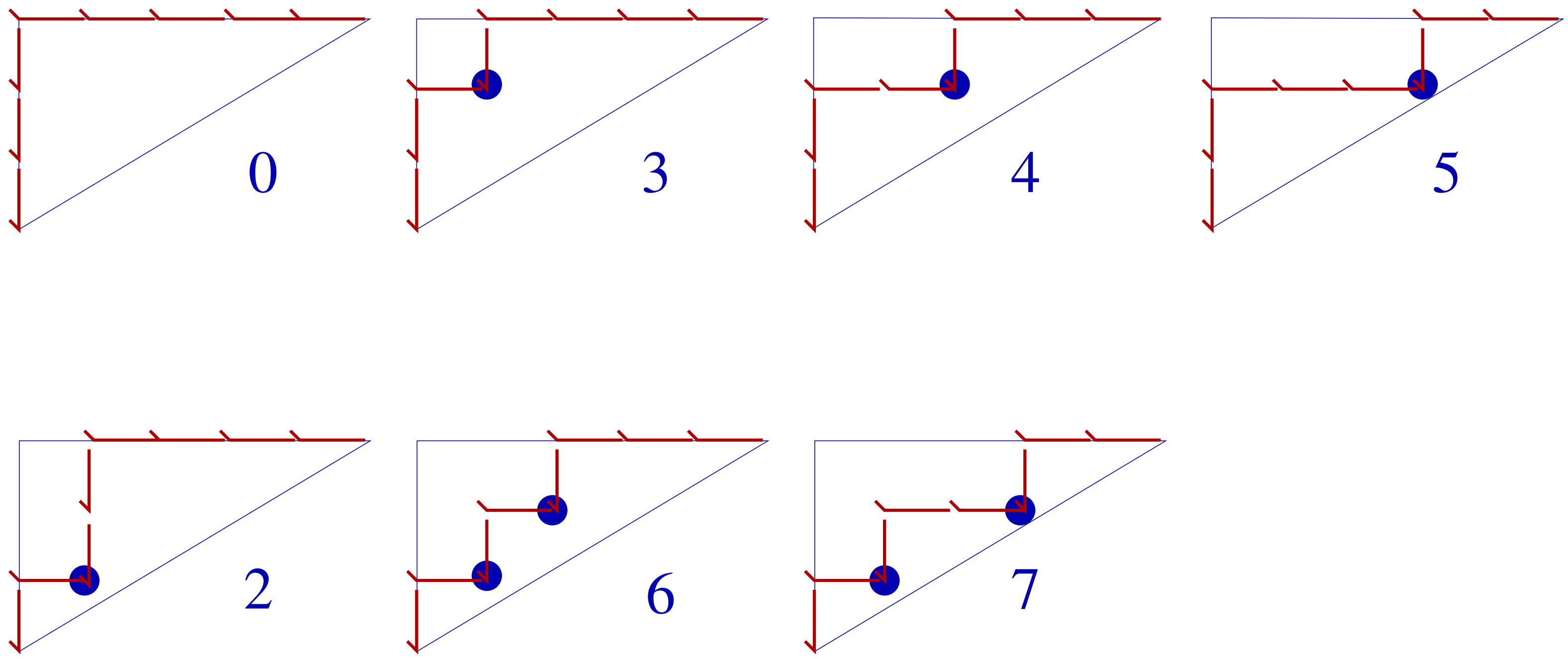
$|\mu|$?



$$\text{Above} = 1 + q + 2q^{q} + 2q^{3} + q^{4},$$

$$\textbf{Cat}_{3,5}(q) = 1 + q^{2} + q^{3} + q^{4} + q^{5} + q^{6} + q^{8}.$$

maj?



$$\text{Above} = 1 + q^2 + q^3 + q^4 + q^5 + q^6 + q^7,$$

$$\mathbf{Cat}_{3,5}(q) = 1 + q^2 + q^3 + q^4 + q^5 + q^6 + q^8.$$

Inspiration: the q, t-Catalan

Garsia and Haiman's rational functions $OC_n(q,t)$ satisfy

$$q^{\binom{n}{2}} OC_n(q, 1/q) = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n, n \end{bmatrix}_q .$$

So let's evaluate the rational q, t-Catalan at $t = 1/q$.

Definitions

Let $A_{\text{max}} = (a-1)(b-1)/2$.

The statistic $h^{+}_{b,a}(\mu)$ is defined to be

$$|\{c \in \mu : -a < a \cdot \textbf{arm}(c) - b \cdot \textbf{leg}(c) \leq b\}|.$$

Examples to follow.

Definitions

The sweep map $\mathbf{sw}$ maps

$$\mathcal{D}^{\mathrm{path}}(N^aE^b) \mapsto \mathcal{D}^{\mathrm{path}}(N^aE^b)$$

$$\mathcal{D}^{\mathrm{ptn}}(a,b) \mapsto \mathcal{D}^{\mathrm{ptn}}(a,b)$$

Fact: $A_{\mathsf{max}} - |\,\mathbf{sw}(\mu)| = h^{+}_{b,a}(\mu)$.

Rational q, t-Catalan

$$\begin{aligned} C_{a,b}(q,t) &= \sum_{w \in \mathcal{D}^{\mathrm{path}}(N^a E^b)} q^{\mathsf{area}(\mathrm{sw}(w))} t^{\mathsf{area}(w)} \\ &= \sum_{\mu \in \mathcal{D}^{\mathrm{ptn}}(a,b)} q^{A_{\mathsf{max}} - |\,\mathrm{sw}(\mu)|} t^{A_{\mathsf{max}} - |\mu|} \\ &= \sum_{\mu \in \mathcal{D}^{\mathrm{ptn}}(a,b)} q^{h^{+}_{b,a}(\mu)} t^{A_{\mathsf{max}} - |\mu|}. \end{aligned}$$

So

$$q^{A_{\mathsf{max}}} C_{a,b}(q, 1/q) = \sum_{\mu \in \mathcal{D}^{\mathrm{ptn}}(a,b)} q^{h^{+}_{b,a}(\mu) + |\mu|}$$

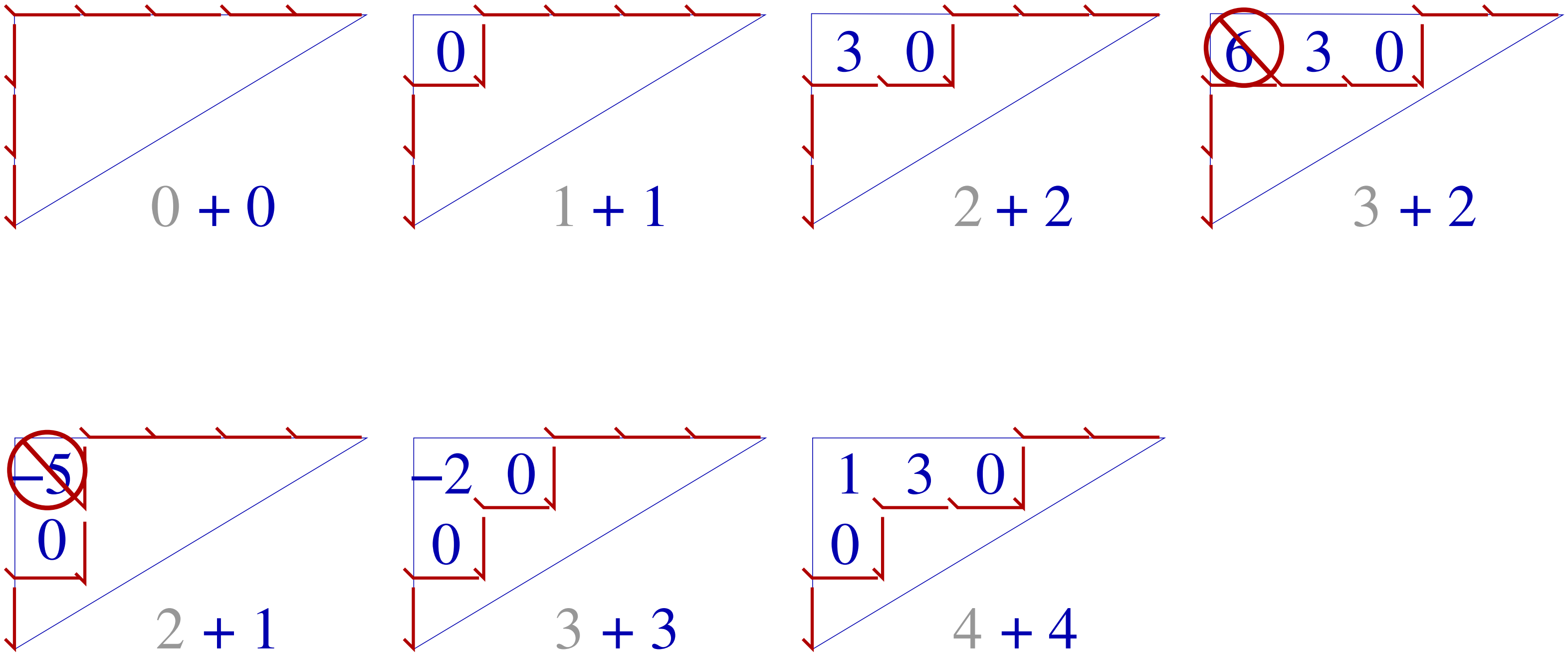
hopefully equals $\textbf{Cat}_{a,b}(q)$.

q-Catalan Conjecture

For $\gcd(a, b) = 1$,

$$\mathsf{Cat}_{a,b}(q) = \sum_{\mu \in \mathcal{D}^{\mathrm{ptn}}(a,b)} q^{|\mu| + h^{+}_{b,a}(\mu)}.$$

Example: $a = 3, b = 5$

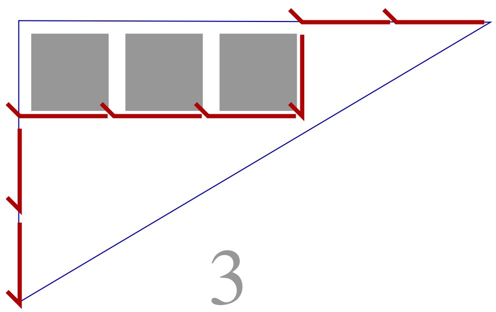


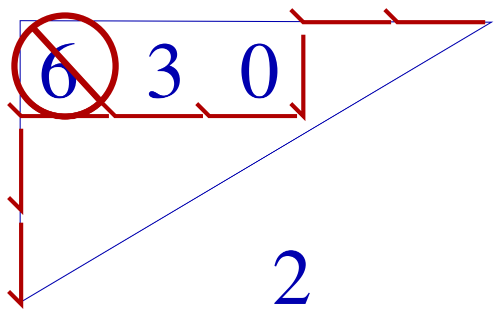
Above $= 1 + q^2 + q^3 + q^4 + q^5 + q^6 + q^8$,

$\mathbf{Cat}_{3,5}(q) = 1 + q^2 + q^3 + q^4 + q^5 + q^6 + q^8$.

Yuck

The same path





contributes q^3 to $\begin{bmatrix} 3+5 \\ 3,5 \end{bmatrix}_q$ but

contributes q^{3+2} to $\mathbf{Cat}_{3,5}(q)$.

Reconciliation

For $\gcd(a,b) = 1$,

$$\mathsf{Cat}_{a,b}(q) = \sum_{\mu \in \mathcal{D}^{\mathrm{ptn}}(a,b)} q^{|\mu| + h^{+}_{b,a}(\mu)}.$$

For $a, b \in \mathbb{N}$, can we find a mysterious, lucky statistic such that

$$\begin{bmatrix} a+b \\ a, b \end{bmatrix}_q = \sum_{\mu \in \mathcal{R}^{\mathrm{ptn}}(a,b)} q^{|\mu| + h^{+}_{b,a}(\mu) + \mathsf{ml}_{b,a}(\mu)}?$$

Tada!

Let $\mathbf{ml}_{b,a}(\mu) = -h^{+}_{b,a}(\mu)$. Then

$$\begin{bmatrix} a+b \\ a,b \end{bmatrix}_q = \sum_{\mu \in \mathcal{R}^{\mathrm{ptn}}(a,b)} q^{|\mu| + h^{+}_{b,a}(\mu) + \mathbf{ml}_{b,a}(\mu)}$$

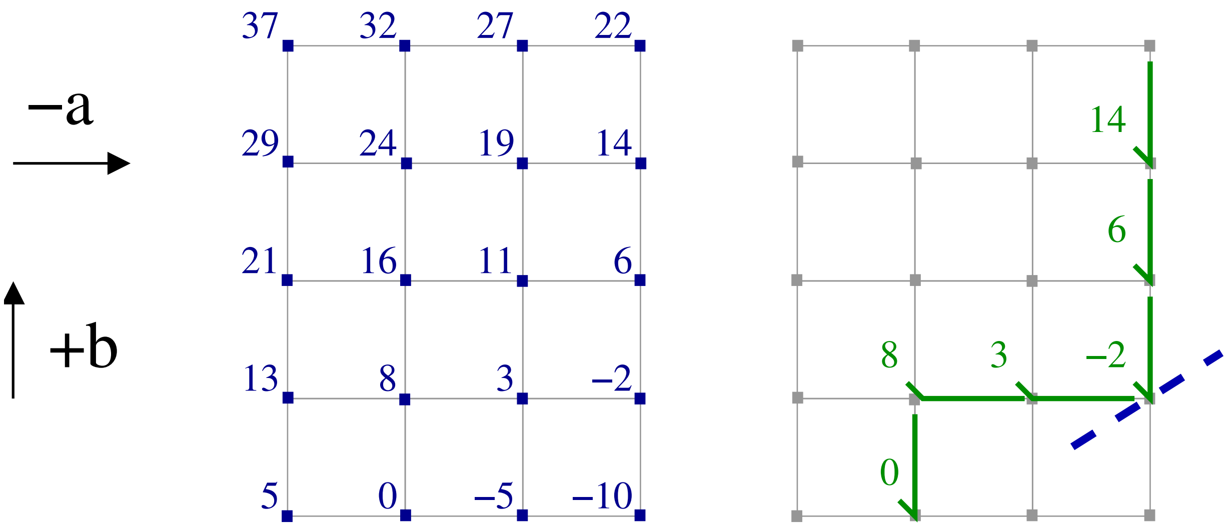
$$= \sum_{\mu \in \mathcal{R}^{\mathrm{ptn}}(a,b)} q^{|\mu|}.$$

So let's require $\mathbf{ml}_{b,a}(\mu) = 0$ when $\mu \in \mathcal{D}^{\mathrm{ptn}}(a,b)$.

Yes

The (a, b)-level of (x, y) is $ay - bx$.

Define $\mathbf{ml}_{b,a}(\mu)$ to be the minimum (a, b)-level of all points on the frontier of μ.

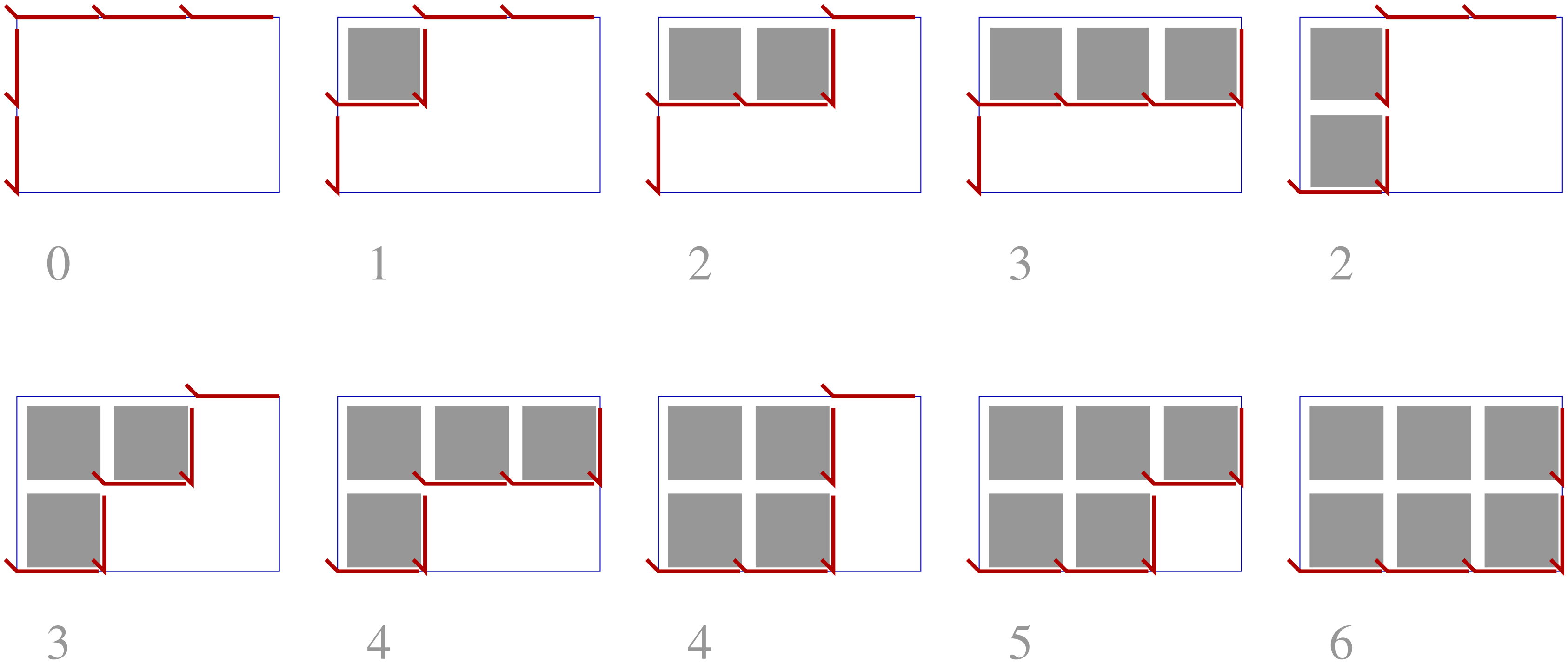


q-Catalan Conjecture

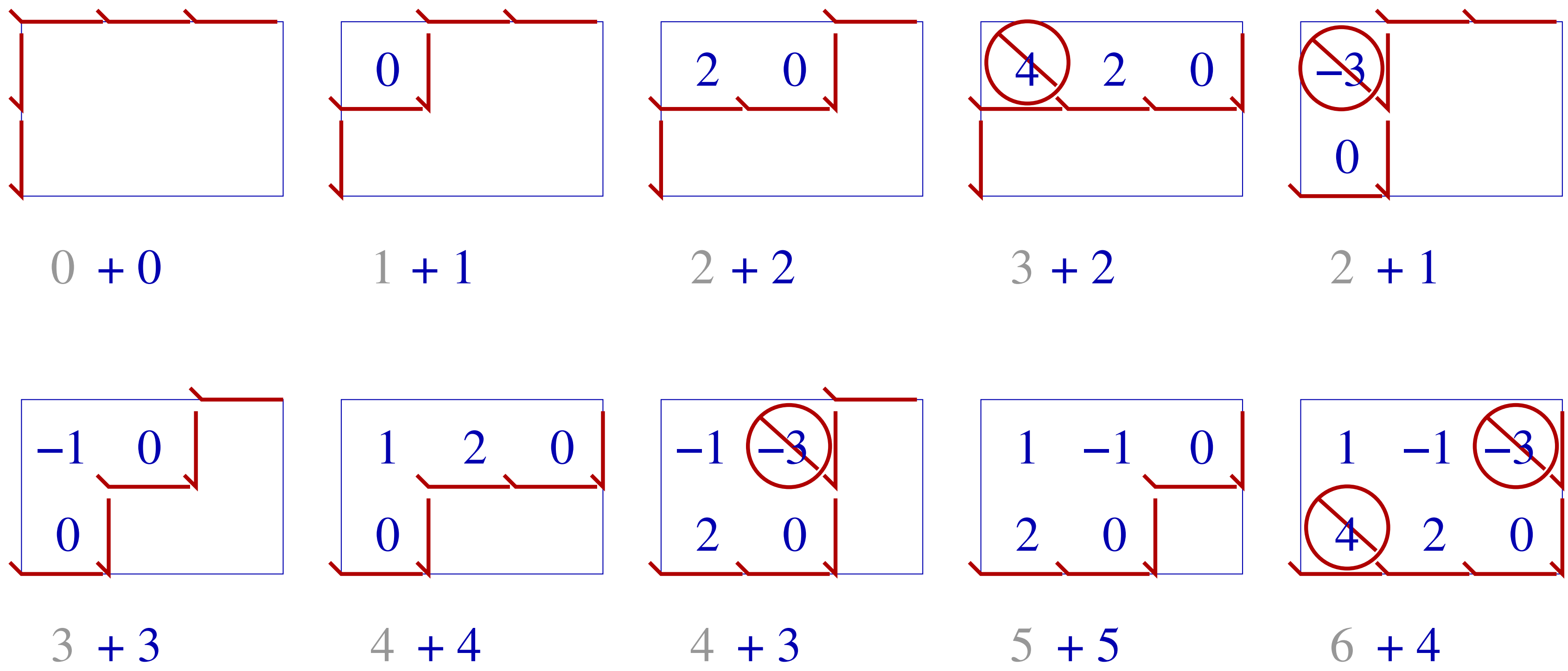
For all $a, b \in \mathbb{Z}_{\geq 0}$, coprime or not,

$$\begin{bmatrix} a+b \\ a,b \end{bmatrix}_q = \sum_{\mu \in \mathcal{D}^{\mathrm{ptn}}(a,b)} q^{|\mu| + \mathsf{ml}_{b,a}(\mu) + h^{+}_{b,a}(\mu)}.$$

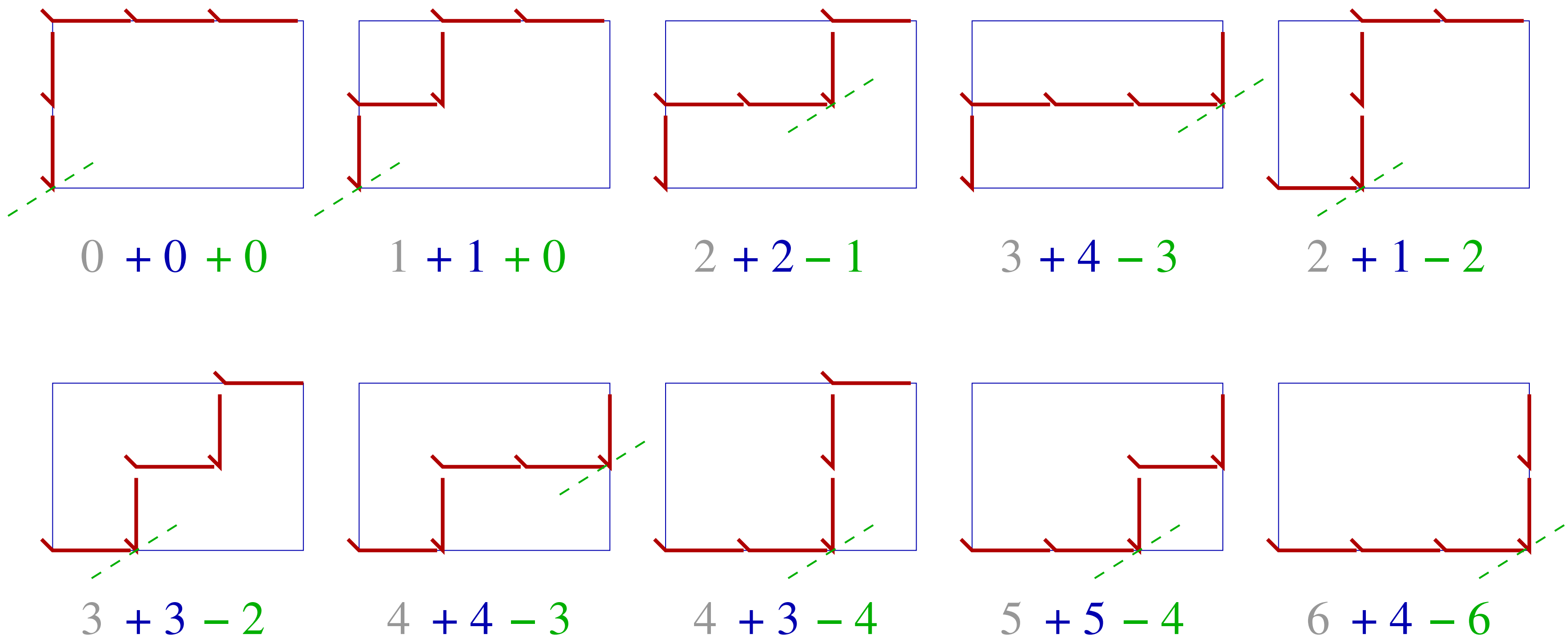
Example: $a = 2, b = 3$



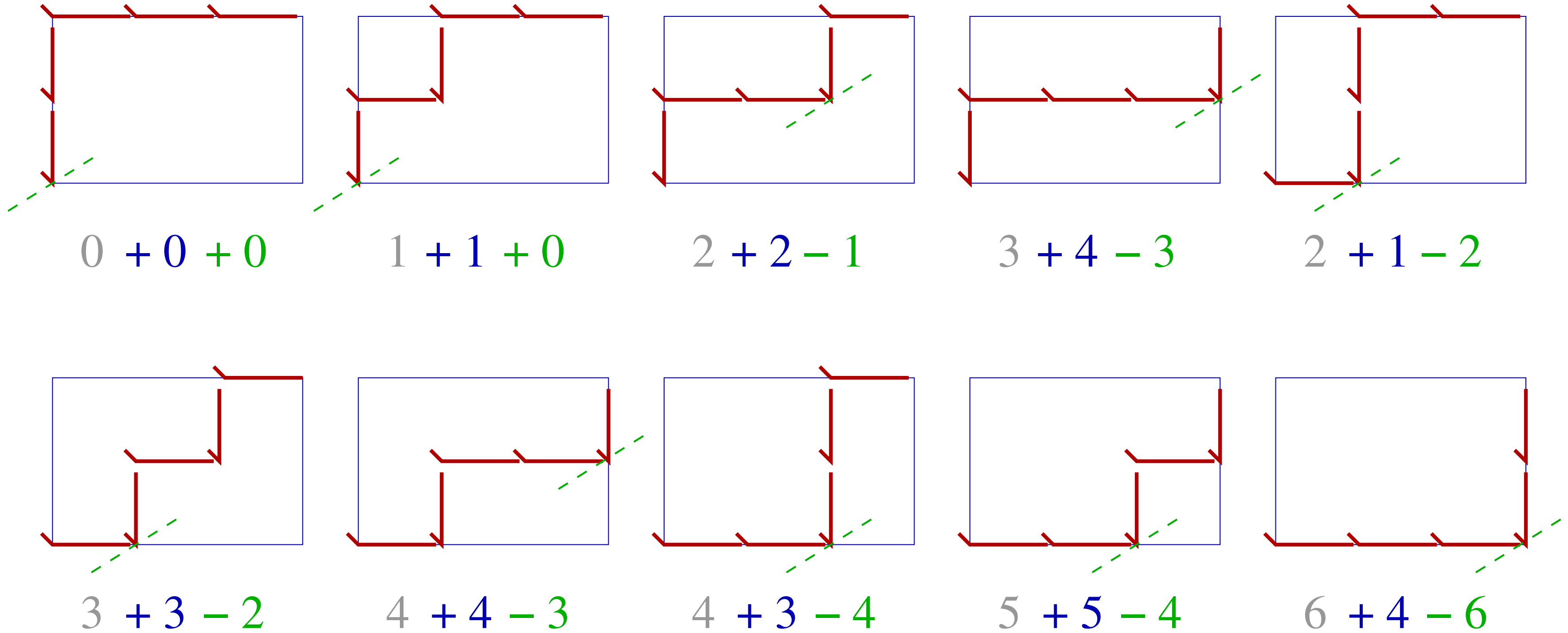
Example: $a = 2, b = 3$



Example: $a = 2, b = 3$



Example: $a = 2, b = 3$



$$1 + q + 2q^2 + 2q^3 + 2q^4 + q^5 + q^6$$

Proof?

Nope.

What we know

Theorem: For $\gcd(a, b) = 1$, the

q-Binomial Conjecture

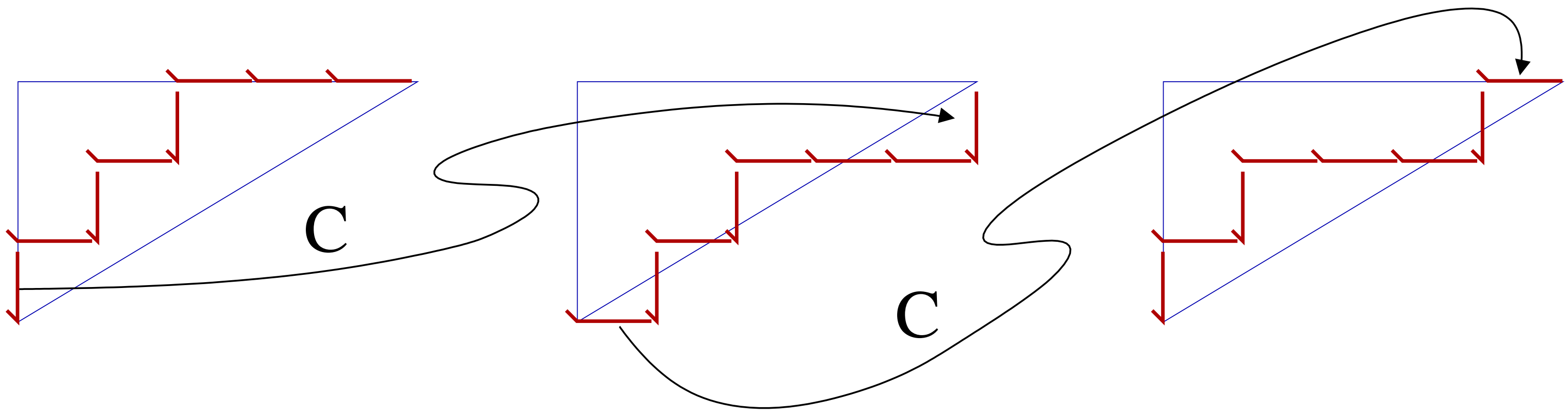
and the

q-Catalan Conjecture

are equivalent.

Sketchy proof

Define a cyclic shift map



Write $\boldsymbol{\mu}^0, \boldsymbol{\mu}^1, \ldots, \boldsymbol{\mu}^{a+b-1}$ for the cyclic shifts of $\boldsymbol{\mu}$.

Claim 1

If $\boldsymbol{\nu}$ is a cyclic shift of $\boldsymbol{\mu} \in \boldsymbol{\mathcal{D}}^{\mathrm{ptn}}(\boldsymbol{a}, \boldsymbol{b})$, then

$$|\boldsymbol{\nu}| + \mathbf{ml}_{b,a}(\boldsymbol{\nu}) = |\boldsymbol{\mu}|.$$

Sketchy proof

$$\begin{bmatrix} a+b \\ a,b \end{bmatrix}_q = \sum_{\mu^0 \in \mathcal{D}^{\mathrm{ptn}}(a,b)} \sum_i q^{|\mu^i| + \mathsf{ml}_{b,a}(\mu^i) + h^+_{b,a}(\mu^i)}$$

$$= \sum_{\mu^0 \in \mathcal{D}^{\mathrm{ptn}}(a,b)} \sum_i q^{|\mu^0| + h^+_{b,a}(\mu^i)}$$

$$= \sum_{\mu^0 \in \mathcal{D}^{\mathrm{ptn}}(a,b)} q^{|\mu^0|} \sum_i q^{h^+_{b,a}(\mu^i)}.$$

Claim 2

If the cyclic shifts of $\boldsymbol{\mu}^0 \in \mathcal{D}^{\mathrm{ptn}}(\boldsymbol{a}, \boldsymbol{b})$ are $\boldsymbol{\mu}^0, \ldots, \boldsymbol{\mu}^{a+b-1}$, then

$$\sum_i \boldsymbol{q}^{h^+_{b,a}(\boldsymbol{\mu}^i)} = [\boldsymbol{a} + \boldsymbol{b}]_q \boldsymbol{q}^{h^+_{b,a}(\boldsymbol{\mu}^0)}.$$

Sketchy proof

$$\begin{aligned}
\begin{bmatrix} a+b \\ a,b \end{bmatrix}_q &= \sum_{\mu^0 \in \mathcal{D}^{\mathrm{ptn}}(a,b)} \sum_i q^{|\mu^i| + \mathsf{ml}_{b,a}(\mu^i) + h^+_{b,a}(\mu^i)} \\
&= \sum_{\mu^0 \in \mathcal{D}^{\mathrm{ptn}}(a,b)} \sum_i q^{|\mu^0| + h^+_{b,a}(\mu^i)} \\
&= \sum_{\mu^0 \in \mathcal{D}^{\mathrm{ptn}}(a,b)} q^{|\mu^0|} \sum_i q^{h^+_{b,a}(\mu^i)} \\
&= \sum_{\mu^0 \in \mathcal{D}^{\mathrm{ptn}}(a,b)} q^{|\mu^0|} [a+b]_q q^{h^+_{b,a}(\mu^0)} \\
&= [a+b]_q \mathsf{Cat}_{a,b}(q).
\end{aligned}$$

Big example