

Quasisymmetric expansions of Schur-function plethysms*

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Abstract

Let s_μ denote a Schur symmetric function and Q_α a fundamental quasisymmetric function. Explicit combinatorial formulas are developed for the fundamental quasisymmetric expansions of the plethysms $s_\mu[s_\nu]$ and $s_\mu[Q_\alpha]$, as well as for related plethysms defined by inequality conditions. The key tools for obtaining these expansions are new standardization and reading word constructions for matrices. As one application, we use our expansions to give a new derivation of the (well-known) Schur expansion for $h_2[h_n]$.

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1 Introduction

For a partition λ , let $|\lambda|$ denote its size and χ^λ the corresponding irreducible character of the symmetric group $S_{|\lambda|}$. There are several natural ways to combine such characters to form new characters of a symmetric group. For example, the *Kronecker* or *inner product* of χ^λ and χ^μ (defined only when $|\lambda| = |\mu|$) is a character of $S_{|\lambda|}$, the *outer product* of χ^λ and χ^μ is a character of $S_{|\lambda|+|\mu|}$, while the *inner plethysm* of χ^λ and χ^μ is a character of $S_{|\lambda||\mu|}$ [19, §I.7,8]. An important general problem in representation theory is to decompose these new characters into irreducible ones.

Using the characteristic map, which sends χ^λ to the Schur function s_λ , this problem can be approached via the theory of symmetric functions. Unfortunately, a fully explicit combinatorial solution is only known for the outer product, where the structure constants can be computed by the Littlewood-Richardson rule. In this paper we restrict our attention to the inner plethysm of s_λ and s_μ , denoted by $s_\lambda[s_\mu]$. The plethysm operation was introduced by Littlewood [16] in the context of compositions of representations of the general linear group. (The modern square-bracket notation reverses the order of his original notation $\{\mu\} \otimes \{\lambda\}$.) Plethysm has important connections to physics [23] and invariant theory [8]. Also, in the form of *plethystic calculus*, it has become a crucial computational tool for understanding diagonal harmonics modules and

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Macdonald’s symmetric functions. One notable example can be found in [10]; see [17] for further references and details regarding plethysm.

Numerous algorithms have been developed for computing the Schur expansion of a plethysm (most recently [5, 6, 24]), and several specific cases and extremal results have been obtained [2, 3, 4, 14, 21, 25]. But an explicit combinatorial formula for the full Schur expansion of an arbitrary plethysm $s_\lambda[s_\mu]$ has proved elusive. This is in sharp contrast to the expansion using *monomial symmetric functions*, which is easily described (see Section 3). In fact, there are several important symmetric functions for which we have combinatorial monomial expansions but lack combinatorially explicit Schur expansions. Examples include the modified Macdonald polynomials [12, 13], Lascoux-Leclerc-Thibon (LLT) polynomials [15] and (conjecturally) the image of a Schur function under the Bergeron-Garsia nabla operator [18].

All of the symmetric functions just mentioned, as well as the Schur functions themselves, have natural combinatorial expansions in terms of Gessel’s *fundamental quasisymmetric functions* Q_α [11]. These formulas arise because, in each case, there is a notion of *standardization* that converts “semistandard” objects appearing in the monomial expansion to “standard” objects. Part of the importance of these quasisymmetric expansions stems from two recently developed methods for converting quasisymmetric expansions of symmetric functions into Schur expansions. Assaf’s method [1] uses a recursive algorithm to create *dual equivalence graphs* whose connected components correspond to terms in the Schur expansion. The second method, due to Egge and the authors [7], uses a modified version of the inverse Kostka matrix to produce explicit combinatorial formulas for the Schur coefficients that involve objects of mixed signs. Of course, for a Schur-positive symmetric function, all the negative objects must somehow cancel with some of the positive objects. The presence of these negative objects is undesirable, but there is significantly less cancellation encountered in converting from a fundamental quasisymmetric expansion to a Schur expansion than in converting from a monomial expansion to a Schur expansion [7].

Our main result is the following explicit combinatorial formula for the expansion of Schur plethysms in terms of the fundamental quasisymmetric functions Q_α .

Theorem 1. *For all partitions μ, ν with $|\mu| = a$ and $|\nu| = b$,*

$$s_\mu[s_\nu] = \sum_{A \in S_{a,b}(\mu, \nu)} Q_{\text{Asc}(A)}. \tag{1}$$

Here, $S_{a,b}(\mu, \nu)$ is a certain set of “standard” $a \times b$ matrices depending on μ and ν , and $\text{Asc}(A)$ is a composition (or subset) determined by a new “reading word” construction for matrices (see Section 4 for precise definitions). Theorem 1 is proved by a standardization bijection on matrices, compatible with the new reading word, that is more subtle than the usual standardization operation on words or tableaux. The same technique furnishes combinatorial formulas for $s_\mu[Q_\alpha]$ and similar plethysms involving polynomials defined by inequality conditions. We note that Malvenuto and Reutenauer [20] have previously proved the Q -positivity of $s_\mu[Q_\alpha]$, a fact that implies, algebraically, the Q -positivity of $s_\mu[s_\nu]$ by plethystic addition rules. In contrast, we give direct bijective proofs of explicit combinatorial formulas for each of these plethysms.

This paper is organized as follows. In Section 2, we review the necessary background on symmetric functions, quasisymmetric functions, and plethysm. Section 3 uses the definition of plethysm to obtain the (well-known) monomial expansion of $s_\mu[s_\nu]$, in which each monomial

arises from a suitably weighted “ μ, ν -matrix.” Section 4 defines ascent sets and reading words for standard matrices. These lead to standardization and unstandardization bijections for matrices. These maps are used in Section 5 to establish Theorem 1 and related formulas. In the final section, we combine these formulas with results from [7] to compute explicitly the (well-known) Schur expansion of $h_2[h_n] = s_{(2)}[s_{(n)}]$. To do this, we introduce a correspondence between standard two-row matrices and certain square lattice paths. Our hope is that this approach may eventually be extended to provide positive combinatorial formulas for Schur expansions of other Schur plethysms.

2 Definitions and Background

An *integer partition* is a weakly decreasing sequence $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k)$ where each $\mu_i \in \mathbb{N}^+$. We say μ is a partition of n , written $\mu \vdash n$, iff $\mu_1 + \cdots + \mu_k = n$. The *diagram* of μ is the set

$$\text{dg}(\mu) = \{(i, j) \in \mathbb{N}^+ \times \mathbb{N}^+ : 1 \leq i \leq k, 1 \leq j \leq \mu_i\}, \quad (2)$$

which can be visualized as a set of unit squares in the first quadrant.

A *semistandard tableau of shape μ* with values in a totally ordered set X is a function $T : \text{dg}(\mu) \rightarrow X$ such that $T(i, j) \leq T(i, j + 1)$ whenever (i, j) and $(i, j + 1)$ are in $\text{dg}(\mu)$, and $T(i, j) < T(i + 1, j)$ whenever (i, j) and $(i + 1, j)$ are in $\text{dg}(\mu)$. The set of all semistandard tableaux of shape μ with values in $\{1, 2, \dots, N\}$ (resp. \mathbb{N}^+) is denoted $\text{SSYT}_N(\mu)$ (resp. $\text{SSYT}(\mu)$). The *content monomial* of a tableau $T \in \text{SSYT}(\mu)$ is $x^{c(T)} = \prod_{(i,j) \in \text{dg}(\mu)} x_{T(i,j)}$. The *Schur symmetric polynomial in N variables indexed by μ* is

$$s_\mu(x_1, \dots, x_N) = \sum_{T \in \text{SSYT}_N(\mu)} x^{c(T)}. \quad (3)$$

The *Schur symmetric function indexed by μ* is the formal series $s_\mu = \sum_{T \in \text{SSYT}(\mu)} x^{c(T)}$. Taking $\mu = (m)$ gives the *complete symmetric polynomial*

$$h_m(x_1, \dots, x_N) = s_{(m)}(x_1, \dots, x_N) = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_m \leq N} x_{i_1} x_{i_2} \cdots x_{i_m}; \quad (4)$$

the complete symmetric function h_m is defined analogously. It is known that the Schur polynomials $s_\mu(x_1, \dots, x_N)$, as μ ranges over integer partitions with at most N parts, form a basis for the \mathbb{Q} -vector space of symmetric polynomials in x_1, \dots, x_N (see, for example, [9]).

Given $k \in \mathbb{N}^+$, a *composition of k* is a sequence $\alpha = (\alpha_1, \dots, \alpha_s)$ of positive integers that sum to k . Compositions of k correspond bijectively to subsets of $\{1, 2, \dots, k - 1\}$ by mapping a composition α to the subset

$$\text{Gap}(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \dots, \alpha_1 + \cdots + \alpha_{s-1}\}.$$

The inverse bijection sends a subset $\{i_1 < i_2 < \cdots < i_s\}$ to the composition $(i_1, i_2 - i_1, \dots, i_s - i_{s-1}, k - i_s)$. In the following, we will sometimes identify the composition α with the subset $\text{Gap}(\alpha)$.

For a composition α of k , Gessel’s *fundamental quasisymmetric polynomial indexed by α* is

$$Q_\alpha(x_1, \dots, x_N) = Q_{\text{Gap}(\alpha)}(x_1, \dots, x_N) = \sum_{\substack{1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq N: \\ \text{strict ascents at} \\ \text{positions in } \text{Gap}(\alpha)}} x_{i_1} x_{i_2} \cdots x_{i_k}. \quad (5)$$

More precisely, the condition on the subscript sequences in the sum is that they be weakly increasing and that $i_j < i_{j+1}$ for all $j \in \text{Gap}(\alpha)$. Let $E(\alpha, N) = E(\text{Gap}(\alpha), N)$ be the set of sequences $\vec{i} = i_1 i_2 \cdots i_k$ satisfying these conditions and let the *weight* of such a sequence be $x_{i_1} \cdots x_{i_k}$. Then $Q_\alpha(x_1, \dots, x_N)$ is the generating function for the weighted set $E(\alpha, N)$. We can define *fundamental quasisymmetric functions* similarly by dropping the requirement that $i_k \leq N$. We let $E(\alpha) = E(\text{Gap}(\alpha))$ denote the associated set of subscript sequences.

Suppose $f(x_1, \dots, x_s)$ and $g(x_1, \dots, x_t)$ are polynomials such that f is symmetric and $g = \sum_{i=1}^s m_i$, where each m_i is a *monic* monomial involving the variables x_1, \dots, x_t . (Note that the number s of variables in f is the same as the number of monomials appearing in the sum for g .) Then we can define the *plethystic substitution of g into f* by setting

$$f[g] = f(m_1, \dots, m_s); \tag{6}$$

the ordering of the monomials m_i is immaterial since f is symmetric. If g is symmetric in x_1, \dots, x_t , so is $f[g]$. The same definition works if f is a symmetric function and g is a formal series that is a sum of countably many monic monomials. The definition of plethysm can be generalized to the case where g is a polynomial or series that may involve negative terms, but we will only need the positive case just defined. See [17] for a more detailed discussion of plethysm.

Suppose $w = w_1 w_2 \cdots w_n$ and $y = y_1 y_2 \cdots y_n$ are two words of the same length using letters in a totally ordered alphabet X . We say that w is *lexicographically less than y* , written $w <_{\text{lex}} y$, iff there exists $k \leq n$ with $w_i = y_i$ for $1 \leq i < k$ and $w_k < y_k$. We write $w \leq_{\text{lex}} y$ iff $w = y$ or $w <_{\text{lex}} y$. One readily verifies that X^n , or any subset thereof, is totally ordered by \leq_{lex} .

3 Monomial Expansion of $s_\mu[s_\nu]$

Informally, $s_\mu[s_\nu]$ is the generating function for semistandard tableaux of shape μ using an alphabet consisting of all semistandard tableaux of shape ν . To make this precise, we encode tableaux using reading words. Given any partition ν , define the *reading order* of the cells of $\text{dg}(\nu)$ by visiting the cells in the shortest row from left to right, then the cells in the next shortest row from left to right, and so on. The *reading word* of a tableau $T \in \text{SSYT}(\nu)$, denoted $\text{rw}(T)$, is

obtained by listing the values $T(i, j)$ in reading order. For example, the tableau $T =$

1	1	2	3
2	2	3	
3	3	4	

of shape $\nu = (4, 3, 3)$ has reading word $\text{rw}(T) = 3342231123$.

If ν is fixed and known, each tableau $T \in \text{SSYT}(\nu)$ is uniquely recoverable from its reading word. We can thereby identify $\text{SSYT}(\nu)$ with the associated set of reading words, denoted $\mathcal{W}(\nu)$. More specifically, given $\nu = (\nu_1, \dots, \nu_s)$, a word $a_1 a_2 \cdots a_{|\nu|}$ is in $\mathcal{W}(\nu)$ iff $a_1 \leq a_2 \leq \cdots \leq a_{\nu_s}$, $a_{\nu_s+1} \leq a_{\nu_s+2} \leq \cdots \leq a_{\nu_s+\nu_{s-1}}$, etc.; and $a_1 > a_{\nu_s+1}$, $a_2 > a_{\nu_s+2}$, etc. The set of words $\mathcal{W}(\nu)$ is totally ordered by \leq_{lex} .

Fix partitions μ and ν with $|\mu| = a$ and $|\nu| = b$. Let $\text{SSYT}_{\mathcal{W}(\nu)}(\mu)$ be the set of semistandard tableaux of shape μ whose entries come from the totally ordered alphabet $\mathcal{W}(\nu)$. We can think of an object $T \in \text{SSYT}_{\mathcal{W}(\nu)}(\mu)$ as a filling of $\text{dg}(\mu)$ where each cell contains a tableau in $\text{SSYT}(\nu)$ (see Figure 1 below). It will also be convenient to encode T as an $a \times b$ matrix A , where row i of A is the reading word of the tableau in the i 'th cell of $\text{dg}(\mu)$ taken in reading order. Each row of A must be a word satisfying the conditions indicated at the end of the last paragraph. In addition, the sequence of rows of A must satisfy analogous conditions with ν replaced by μ . For instance, if $\mu = (\mu_1, \dots, \mu_t)$, the first μ_t rows of A must form a lexicographically increasing

sequence of words; row 1 of A must be lex greater than row $\mu_t + 1$ of A ; etc. Matrices A satisfying these conditions will be called μ, ν -matrices; let $M_{a,b}(\mu, \nu)$ be the set of all such matrices. The *weight* of $A \in M_{a,b}(\mu, \nu)$, denoted $\text{wt}(A)$, is the product $\prod_{i,j} x_{A(i,j)}$. Combining (3) and (6), we deduce

$$s_\mu[s_\nu] = \sum_{A \in M_{a,b}(\mu, \nu)} \text{wt}(A). \quad (7)$$

Example 2. Let $\mu = (2, 1)$ and $\nu = (2, 2)$. The μ, ν -matrix $C = \begin{bmatrix} 4 & 5 & 3 & 4 \\ 3 & 5 & 1 & 3 \\ 4 & 6 & 2 & 2 \end{bmatrix}$ contributes one term $x_1 x_2^2 x_3^3 x_4^3 x_5^2 x_6$ to $s_{(2,1)}[s_{(2,2)}]$. C encodes the “nested tableau” shown in Figure 1.

1	3	2	2
3	5	4	6
3		4	
4		5	

Figure 1: Nested tableau corresponding to one term of $s_{(2,1)}[s_{(2,2)}]$.

4 Ascent Sets and Standardization for Matrices

To convert the monomial expansion in the last section to the quasisymmetric expansion of Theorem 1, we need to define notions of ascent sets and standardization for matrices. First, we review the more elementary versions of these concepts that apply to words and tableaux.

4.1 Standardization of Words

For $n \in \mathbb{N}^+$, let S_n be the set of permutations of $\{1, 2, \dots, n\}$, i.e., words in which each symbol between 1 and n occurs exactly once. Given any word $v = v_1 v_2 \cdots v_n$ with each $v_i \in \mathbb{N}^+$, we can produce its *standardization* $\text{std}(v) \in S_n$ as follows. Suppose v contains k_1 ones, k_2 twos, etc. Replace the ones in v from left to right with $1, 2, \dots, k_1$; then replace the twos in v from left to right with $k_1 + 1, k_1 + 2, \dots, k_1 + k_2$; and so on. Also, let $\text{sort}(v)$ be the word obtained by sorting the letters of v into weakly increasing order, and let $\text{wt}(v) = \prod_{i=1}^n x_{v_i} = x_1^{k_1} x_2^{k_2} \cdots$. For example, given $v = 311422413$, we have

$$\text{std}(v) = 612845937 \in S_9; \quad \text{sort}(v) = 111223344; \quad \text{wt}(v) = \text{wt}(\text{sort}(v)) = x_1^3 x_2^2 x_3^2 x_4^2.$$

Given $w \in S_n$, the *descent set* of w is $\text{Des}(w) = \{i < n : w_i > w_{i+1}\}$. The *inverse descent set* of w is $\text{IDes}(w) = \text{Des}(w^{-1})$, which consists of all $j < n$ such that $j+1$ appears somewhere to the left of j in w . We can identify these subsets of $\{1, 2, \dots, n-1\}$ with compositions of n using the bijection in the introduction. For example, given $w = 612845937$, we have $\text{Des}(w) = \{1, 4, 7\}$ and $\text{Des}(w^{-1}) = \{3, 5, 7\}$. These subsets correspond to the compositions $(1, 3, 3, 2)$ and $(3, 2, 2, 2)$, respectively.

Let X be the set of words of length n in the alphabet \mathbb{N}^+ . Let Y be the set of pairs (w, \vec{i}) , where $w \in S_n$ and $\vec{i} = i_1 i_2 \cdots i_n$ is a subscript sequence in $E(\text{IDes}(w))$. One may check that the

map $v \mapsto (\text{std}(v), \text{sort}(v))$ defines a bijection of X onto Y . This bijection shows that

$$\sum_{v \in X} \text{wt}(v) = \sum_{w \in S_n} Q_{\text{IDes}(w)}.$$

In turn, using well-known properties of the Robinson-Schensted-Knuth (RSK) algorithm and the Q -expansion of s_λ (implicit in [11]), one can verify that

$$\sum_{w \in S_n} Q_{\text{IDes}(w)} = \sum_{\lambda \vdash n} f^\lambda s_\lambda, \quad (8)$$

where f^λ is the number of standard tableaux of shape λ . This symmetric function is the Frobenius character of the left regular representation of the symmetric group S_n .

4.2 Standard Matrices and Ascent Sets

Let $M_{a,b}$ be the set of all $a \times b$ matrices with entries in \mathbb{N}^+ . A matrix in $M_{a,b}$ is called *standard* iff its entries are $1, 2, \dots, ab$ in some order. Let $S_{a,b}$ be the set of standard $a \times b$ matrices. Given a standard matrix $A \in S_{a,b}$, we define its *reading word* $\text{rw}(A)$ as follows. Scan the columns of A from left to right. For each column index $k < b$, use the symbols in column $k + 1$ of A to determine a total ordering of the rows in which the row with the smallest symbol in column $k + 1$ comes first, and the row with the largest symbol in column $k + 1$ comes last. Write the symbols in column k using the row ordering just determined. For the rightmost column, write the symbols in order from top to bottom. Given a and b , we can recover a standard matrix A from its reading word $\text{rw}(A)$ by scanning the word backwards to recreate the columns of A from right to left. Thus, the map $\text{rw} : S_{a,b} \rightarrow S_{ab}$ is a bijection.

The *ascent set* of the matrix A is $\text{Asc}(A) = \text{IDes}(\text{rw}(A))$. So $j \in \text{Asc}(A)$ iff $j + 1$ appears to the left of j in $\text{rw}(A)$. One may check that $j \in \text{Asc}(A)$ iff either (a) $j + 1$ appears in A in a column to the left of the column containing j ; or (b) j and $j + 1$ are both in the rightmost column of A , with $j + 1$ in a higher row than j ; or (c) j and $j + 1$ are in the same column (but not the rightmost column), and the number to the right of j in its row exceeds the number to the right of $j + 1$ in its row.

Example 3. The matrix

$$A = \begin{bmatrix} 8 & 7 & 2 & 10 \\ 6 & 1 & 9 & 4 \\ 12 & 5 & 11 & 3 \end{bmatrix} \in S_{3,4}$$

has reading word $\text{rw}(A) = 612871511921043$ and ascent set $\text{Asc}(A) = \{3, 4, 5, 7, 10, 11\}$. Given that $B \in S_{3,4}$ has reading word $\text{rw}(B) = 412869105273111$, we work backwards through the reading word to find that

$$B = \begin{bmatrix} 4 & 6 & 2 & 3 \\ 8 & 10 & 7 & 11 \\ 12 & 9 & 5 & 1 \end{bmatrix}.$$

4.3 Standardization of Matrices

Fix a matrix $C \in M_{a,b}$. Define the *weight* of C to be $\text{wt}(C) = \prod_{i=1}^a \prod_{j=1}^b x_{C(i,j)}$ and $\text{sort}(C)$ to be the word $w_1 w_2 \cdots w_{ab}$ obtained by listing the entries of C in weakly increasing order. Define $N_{a,b}$ to be the set of pairs (A, \vec{i}) for which $A \in S_{a,b}$ and $\vec{i} = i_1 i_2 \cdots i_{ab} \in E(\text{Asc}(A))$. Let the weight of such a pair be $\text{wt}(A, \vec{i}) = \text{wt}(\vec{i}) = \prod_{k=1}^{ab} x_{i_k}$.

Theorem 4. *For all $a, b \in \mathbb{N}^+$, there are weight-preserving, mutually inverse bijections $S : M_{a,b} \rightarrow N_{a,b}$ (standardization) and $U : N_{a,b} \rightarrow M_{a,b}$ (unstandardization). Moreover,*

$$\sum_{C \in M_{a,b}} \text{wt}(C) = \sum_{A \in S_{a,b}} Q_{\text{Asc}(A)} = \sum_{w \in S_{ab}} Q_{\text{IDes}(w)} = \sum_{\lambda \vdash ab} f^\lambda s_\lambda. \quad (9)$$

Proof. Given a pair $(A, \vec{i}) \in N_{a,b}$, we compute its *unstandardization* $U(A, \vec{i}) \in M_{a,b}$ by replacing the unique copy of j in A by i_j , for $1 \leq j \leq ab$. For example,

$$U \left(\left(\begin{bmatrix} 1 & 4 & 6 \\ 2 & 5 & 12 \\ 3 & 9 & 11 \\ 7 & 8 & 10 \end{bmatrix}, 1111111222234 \right) \right) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 4 \\ 1 & 2 & 3 \\ 2 & 2 & 2 \end{bmatrix}.$$

Given a matrix $C \in M_{a,b}$, its *standardization* is $S(C) = (\text{std}(C), \text{sort}(C))$, where $\text{std}(C)$ is the standard matrix obtained by the following algorithm. For each $k \geq 0$, let $N(k)$ be the number of symbols in C that are less than or equal to k . Note that $N(0) = 0$ and $N(k) = ab$ for large enough k . For any $i > 0$, there are $N(i) - N(i-1)$ occurrences of i in C . These i 's will get renumbered by *standard labels* in the set $L_i = \{N(i-1) + 1, N(i-1) + 2, \dots, N(i)\}$. The standardization algorithm operates as follows:

1. Scan the rightmost column of C from bottom to top, replacing each symbol i as it is encountered by the largest unused label in L_i .
2. For each j from $b-1$ down to 1, renumber the entries in column j as follows. The previously renumbered column $j+1$ determines a total ordering of the rows, as in the definition of the ascent set. Scan the entries in column j (according to the row ordering determined by column $j+1$) from the largest row to the smallest row. Replace each symbol i as it is encountered by the largest unused label in L_i .

Example 5. An example of the standardization process is shown here:

$$C = \begin{bmatrix} 1 & 1 & 3 & 3 & 5 \\ 1 & 2 & 2 & 2 & 4 \\ 2 & 2 & 3 & 3 & 3 \end{bmatrix} \xrightarrow{S} \left(A = \begin{bmatrix} 1 & 3 & 10 & 12 & 15 \\ 2 & 5 & 7 & 8 & 14 \\ 4 & 6 & 9 & 11 & 13 \end{bmatrix}, 111222223333345 \right). \quad (10)$$

Note that $N(0) = 0$, $N(1) = 3$, $N(2) = 8$, $N(3) = 13$, $N(4) = 14$, and $N(5) = 15$, so that $L_1 = \{1, 2, 3\}$, $L_2 = \{4, 5, 6, 7, 8\}$, $L_3 = \{9, 10, 11, 12, 13\}$, $L_4 = \{14\}$, and $L_5 = \{15\}$. The columns of A are filled in from right to left, with the scanning order for each column determined by the next column to the right. The last column is scanned from bottom to top; the fourth column from top to bottom; the third and second columns both in the order row 1, row 3, then row 2; and the first column from bottom to top. We have $\text{rw}(A) = 124563791011812151413$ and $\text{Asc}(A) = \{3, 8, 13, 14\}$. Observe that $S(C) \in N_{3,5}$ and $U(S(C)) = C$.

Continuing the proof, we must show that $S(C)$ is always in $N_{a,b}$. Since $A = \text{std}(C) \in S_{a,b}$, it is enough to verify that $\text{sort}(C) \in E(\text{Asc}(A))$. Write $\text{sort}(C) = i_1 i_2 \cdots i_{ab}$. This sequence is weakly increasing by definition. We see (by backwards induction on the column index) that the standardization algorithm creates the symbols in A in the same order as they occur in a backwards scan of $\text{rw}(A)$. Since the labels in each set L_i are chosen from largest to smallest during this process, it follows that the subsequence of $\text{rw}(A)$ consisting of the symbols in L_i (read from left to right) is weakly increasing. Therefore, $\text{Asc}(A) \subseteq \{N(1), N(2), \dots\}$. Now, if $N(i-1) < k \leq N(i)$, then the symbol k in A was used to renumber one of the i 's in C , and so $i_k = i$. Therefore, $k \in \text{Asc}(A)$ implies $i_k < i_{k+1}$, proving that $\text{sort}(C) \in E(\text{Asc}(A))$.

One may routinely verify that $U \circ S = \text{id}_{M_{a,b}}$. It is a little trickier to see that $S \circ U = \text{id}_{N_{a,b}}$. For this, fix $x = (A, i_1 \cdots i_{ab}) \in N_{a,b}$, and set $C = U(x) \in M_{a,b}$, $x' = S(C) = (A', i'_1 \cdots i'_{ab}) \in N_{a,b}$. Evidently, $i_1 \cdots i_{ab} = \text{sort}(C) = i'_1 \cdots i'_{ab}$. We must argue that $A' = A$. Let $N(i)$ and L_i be determined from C as in the definition of $\text{std}(C)$. When passing from A to C , all labels in L_i get replaced by occurrences of i . In turn, when passing from C to A' , these occurrences of i get replaced by the standard labels in L_i . The key point to check is that this second replacement puts each $k \in L_i$ back in the same position in A' that k occupied in A . The rules defining S and U are designed to force this property to hold. More precisely, in both of the reading words $\text{rw}(A)$ and $\text{rw}(A')$, the labels in each L_i must occur as an increasing subsequence; this holds for $\text{rw}(A)$ since $i_1 \cdots i_{ab} \in E(\text{Asc}(A))$, and it holds for $\text{rw}(A')$ as shown in the previous paragraph. We can then prove that $A = A'$ by backwards induction on the column index.

Thus U and S are inverses of each other, so both are weight-preserving bijections. Hence

$$\sum_{C \in M_{a,b}} \text{wt}(C) = \sum_{A \in S_{a,b}} \left(\sum_{\vec{i} \in E(\text{Asc}(A))} \text{wt}(\vec{i}) \right). \quad (11)$$

The inner sum is precisely $Q_{\text{Asc}(A)} = Q_{\text{IDes}(\text{rw}(A))}$. Since $\text{rw} : S_{a,b} \rightarrow S_{ab}$ is a bijection, (9) follows from (11) and (8). \square

5 Proof of Theorem 1

Given partitions $\mu \vdash a$ and $\nu \vdash b$, recall from Section 3 that $M_{a,b}(\mu, \nu)$ is the subset of $M_{a,b}$ consisting of μ, ν -matrices (which encode tableaux in $\text{SSYT}_{\mathcal{W}(\nu)}(\mu)$). Let $S_{a,b}(\mu, \nu) = S_{a,b} \cap M_{a,b}(\mu, \nu)$ be the set of standard μ, ν -matrices. Let $N_{a,b}(\mu, \nu)$ consist of all pairs $(A, \vec{i}) \in N_{a,b}$ with $A \in S_{a,b}(\mu, \nu)$. Our goal is to prove Theorem 1, which states

$$s_\mu[s_\nu] = \sum_{A \in S_{a,b}(\mu, \nu)} Q_{\text{Asc}(A)}.$$

In light of Theorem 4, this equation will follow from (7) once we show that the maps S and U restrict to give bijections between $M_{a,b}(\mu, \nu)$ and $N_{a,b}(\mu, \nu)$. This amounts to showing that $C \in M_{a,b}(\mu, \nu)$ iff $\text{std}(C) \in S_{a,b}(\mu, \nu)$. Keeping in mind the definition of μ, ν -matrices, this last fact is an immediate consequence of the order-preservation properties appearing in the following lemma.

Lemma 6. *Let $C \in M_{a,b}$ and $A = \text{std}(C)$.*

- (a) *For $1 \leq i \leq a$ and $1 \leq j < k \leq b$, $C(i, j) \leq C(i, k)$ iff $A(i, j) < A(i, k)$, and $C(i, j) > C(i, k)$ iff $A(i, j) > A(i, k)$.*

(b) Let C_i (resp. A_i) be row i of C (resp. A), viewed as a word of length b . For $1 \leq i < j \leq a$, $C_i \leq_{\text{lex}} C_j$ iff $A_i <_{\text{lex}} A_j$, and $C_j <_{\text{lex}} C_i$ iff $A_j <_{\text{lex}} A_i$.

Proof. It suffices to prove the forward directions of the four biconditional statements.

(a) Let $x = C(i, j)$ and $y = C(i, k)$. First, $x > y$ implies $A(i, j) > A(i, k)$ since every label in L_x exceeds every label in L_y . Second, $x < y$ implies $A(i, j) < A(i, k)$ for the same reason. Third, $x = y$ implies $A(i, j) < A(i, k)$ since standardization creates the columns of A from right to left, with occurrences of x in C replaced by elements of L_x in decreasing order.

(b) Fix $i < j$ and assume $C_i <_{\text{lex}} C_j$. This means there exists $k \leq b$ with $C(i, s) = C(j, s)$ for $1 \leq s < k$, and $C(i, k) < C(j, k)$. The last condition implies that $A(i, k) < A(j, k)$. It then follows that $A(i, s) < A(j, s)$ for $1 \leq s \leq k$ by backwards induction on s , keeping in mind the standardization rules for two equal labels in the same column. In particular, $A(i, 1) < A(j, 1)$, so that $A_i <_{\text{lex}} A_j$. The same argument shows that $C_j <_{\text{lex}} C_i$ implies $A_j <_{\text{lex}} A_i$. Finally, consider the case $C_i = C_j$. Since $i < j$, $A(i, b) < A(j, b)$ follows by the standardization rule for the rightmost column. Then $A(i, s) < A(j, s)$ follows for $1 \leq s \leq b$ by backwards induction on s , as before. So $A(i, 1) < A(j, 1)$ and $A_i <_{\text{lex}} A_j$. \square

Example 7. Take $\mu = (2, 1)$ and $\nu = (2, 2)$, and let $C = \begin{bmatrix} 4 & 5 & 3 & 4 \\ 3 & 5 & 1 & 3 \\ 4 & 6 & 2 & 2 \end{bmatrix}$ as in Example 2.

We compute $A = \text{std}(C) = \begin{bmatrix} 7 & 11 & 5 & 9 \\ 4 & 10 & 1 & 6 \\ 8 & 12 & 2 & 3 \end{bmatrix}$ and $S(C) = (A, 122333444556) \in N_{3,4}(\mu, \nu)$.

Furthermore, $U(A, 122344556778) = \begin{bmatrix} 5 & 7 & 4 & 6 \\ 3 & 7 & 1 & 4 \\ 5 & 8 & 2 & 2 \end{bmatrix} \in M_{3,4}(\mu, \nu)$.

Example 8. Now take $\mu = (2, 2)$ and $\nu = (2, 1)$. We compute

$$S \left(\begin{pmatrix} \begin{bmatrix} 2 & 1 & 2 \\ 3 & 1 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \end{pmatrix} \right) = \left(\begin{pmatrix} \begin{bmatrix} 9 & 3 & 10 \\ 12 & 4 & 11 \\ 7 & 1 & 5 \\ 8 & 2 & 6 \end{bmatrix}, 111111222223 \end{pmatrix} \right),$$

where both displayed matrices are μ, ν -matrices.

Example 9. Theorem 1 and the data in Table 1 lead to the formula

$$h_2[h_3] = s_{(2)}[s_{(3)}] = Q_{33} + Q_{222} + Q_{24} + Q_{231} + Q_{42} + Q_6 + Q_{321} + Q_{123} + Q_{141} + Q_{132}.$$

Lemma 6 and matrix standardization provide the Q -expansions of many other plethysms, analogous to $s_\mu[s_\nu]$, whose monomial expansions are characterized by appropriately chosen inequalities. For instance, given $\mu \vdash a$ and $T \subseteq \{1, 2, \dots, b-1\}$ (corresponding to some composition α of b), we can obtain the quasisymmetric expansion of $s_\mu[Q_\alpha]$ as follows. Recall that Q_α is the generating function for weakly increasing sequences $\vec{i} = i_1 i_2 \cdots i_b$ with strict ascents at positions in T . Write $T = \{t_1 < \cdots < t_k\}$, and define the *reading word* of \vec{i} to be

$$\text{rw}(\vec{i}) = i_{t_k+1} i_{t_k+2} \cdots i_b i_{t_{k-1}+1} i_{t_{k-1}+2} \cdots i_{t_k} \cdots i_1 i_2 \cdots i_{t_1}.$$

Table 1: Q -expansion of $h_2[h_3]$.

A	$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$	$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 5 & 6 \end{bmatrix}$	$\begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{bmatrix}$	$\begin{bmatrix} 1 & 2 & 6 \\ 3 & 4 & 5 \end{bmatrix}$	$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 5 & 6 \end{bmatrix}$
$\text{rw}(A)$	142536	132546	132456	134265	123546
α	33	222	24	231	42
A	$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$	$\begin{bmatrix} 1 & 3 & 6 \\ 2 & 4 & 5 \end{bmatrix}$	$\begin{bmatrix} 1 & 4 & 5 \\ 2 & 3 & 6 \end{bmatrix}$	$\begin{bmatrix} 1 & 4 & 6 \\ 2 & 3 & 5 \end{bmatrix}$	$\begin{bmatrix} 1 & 5 & 6 \\ 2 & 3 & 4 \end{bmatrix}$
$\text{rw}(A)$	123456	124365	214356	213465	213564
α	6	321	123	141	132

Let $E'(T) = E'(\alpha)$ be the set of such reading words. One sees that membership in $E'(T)$ is characterized by inequalities of the type appearing in Lemma 6(a). For example, $w = w_1 \cdots w_9 \in E'(3213)$ iff $w_1 \leq w_2 \leq w_3$ and $w_5 \leq w_6$ and $w_7 \leq w_8 \leq w_9$ and $w_1 > w_4 > w_6$ and $w_5 > w_9$. Now let $M_{a,b}(\mu, T)$ be the set of $a \times b$ matrices where each row is in $E'(T)$, and the sequence of rows is the reading word of a tableau in $\text{SSYT}_{E'(T)}(\mu)$. Then (6) shows that $s_\mu[Q_\alpha] = \sum_{C \in M_{a,b}(\mu, T)} \text{wt}(C)$. Define $S_{a,b}(\mu, T) = S_{a,b} \cap M_{a,b}(\mu, T)$ and $N_{a,b}(\mu, T) = \{(A, \vec{v}) : A \in S_{a,b}(\mu, T)\}$. Lemma 6 and matrix standardization show that

$$s_\mu[Q_\alpha] = \sum_{A \in S_{a,b}(\mu, T)} Q_{\text{Asc}(A)},$$

as in the proof of Theorem 1. Other plethysms $f[g]$, where f and g are “defined by \leq and $>$ conditions,” can be handled in an entirely analogous way.

6 Schur expansion of $h_2[h_n]$

A closed formula for the plethysm $h_2[h_n]$ can be computed algebraically (see, for example, [19, I.8, Example 9]). In fact, such a formula was already known to Thrall [22]. Here we give a new combinatorial derivation as an illustration of how one can recover a Schur expansion from our Q_α expansions.

The matrices in $M_{2,n}((2), (n))$ are those for which each row is weakly increasing and the first row occurs weakly before the second lexicographically. Given $A \in S_{2,n}((2), (n))$, we define a word $p(A) = p_1 \cdots p_{2n} \in \{0, 1\}^{2n}$ by setting $p_i = 0$ if i occurs in the first row of A and $p_i = 1$ if i occurs in the second row. The word $p(A)$ encodes a lattice path $v_0 \cdots v_{2n}$ from $v_0 = (0, 0)$ to $v_{2n} = (n, n)$ by associating each 0 with a unit-length north step and each 1 with a unit-length east step. We use this correspondence without further comment.

Define the *level* of each vertex v_i on the path $p(A)$ to be the value $\ell(v_i)$ for which v_i lies on the line $y = x + \ell(v_i)$. By convention, let $\ell(v_{2n+1}) = 1$. For $0 < i < 2n$, say that vertex v_i is *marked* if one of the following conditions holds:

- $|\ell(v_i)| > |\ell(v_{i+1})| > 0$.
- $\ell(v_{i-1}) = 0$ and $\ell(v_i) = -\ell(v_{i+2}) = \pm 1$.

The first condition marks right-turns that occur strictly above $y = x + 1$ and left-turns that occur strictly below $y = x - 1$. The second condition marks each turn occurring one step before the path crosses the line $y = x$. When using this condition to decide whether v_{2n-1} should be marked, one imagines that the path continues north after the point (n, n) . We refer to a path $p(A)$ along with its marked vertices as a *marked lattice path*. Let $m(A) = (m_1(A), m_2(A), \dots, m_k(A))$ be the sequence of marked points of the path $p(A)$. In Figure 2, we display two elements of $S_{2,7}((2), (7))$ along with their associated marked lattice paths.

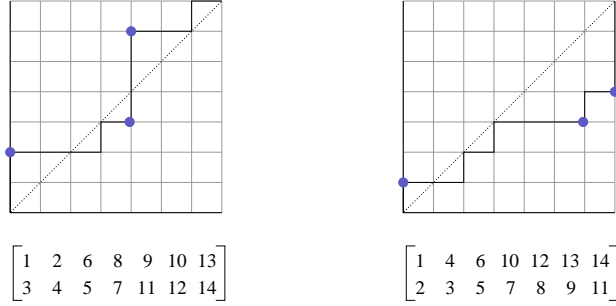


Figure 2: Marked lattice paths.

If $m(A) = (v_{i_1}, \dots, v_{i_k})$, one readily checks that $\text{Asc}(A) = \{i_1, \dots, i_k\}$. For the rest of this section, we will use the bijection in the introduction to allow us to view $\text{Asc}(A)$ as a composition instead of a subset. For example, the matrix A shown on the left in Figure 2 has $\text{Asc}(A) = (2, 5, 3, 4)$.

Theorem 10. *For all $n \geq 1$,*

$$h_2[h_n] = \sum_{\substack{c=0 \\ c \text{ even}}}^n s_{(2n-c,c)}. \quad (12)$$

Proof. We use the procedure for converting Q -expansions to Schur expansions described in [7]. In general, given a symmetric function f with Q -expansion $f = \sum_{\alpha} y_{\alpha} Q_{\alpha}$ and Schur expansion $f = \sum_{\lambda} x_{\lambda} s_{\lambda}$, each x_{λ} is given by a signed sum of the y_{α} indexed by those α for which there exists a *flat special rim-hook tableau* of shape λ and content α . More specifically, the following facts follow immediately from results in [7]:

1. $x_{(2n)} = y_{(2n)}$.
2. For $0 < c \leq n$, $x_{(2n-c,c)} = y_{(2n-c,c)} - y_{(c-1, 2n-c+1)}$.
3. Suppose that for all $\alpha = (\alpha_1, \alpha_2, \alpha_3, \dots)$ with $\alpha_1 \geq \alpha_2$, we have $y_{\alpha} = y_{(\alpha_2-1, \alpha_1+1, \alpha_3, \dots)}$. Then $x_{\lambda} = 0$ whenever λ is a partition with at least three parts.

We apply these facts to $f = h_2[h_n]$. First, we claim $x_{(2n)} = 1$. Now, $y_{(2n)} = 1$ since the only $A \in S_{2,n}((2), (n))$ with $\text{Asc}(A) = (2n)$ is $A = \begin{bmatrix} 1 & 3 & 5 & \cdots \\ 2 & 4 & 6 & \cdots \end{bmatrix}$. The claim then follows by Fact 1. Next, for $0 < c \leq n$, we claim that $x_{(2n-c,c)}$ is 1 if c is even and 0 if c is odd. This is a consequence of Fact 2 and a routine counting argument, left to the reader. See Figure 3 for examples of both the even and odd cases. The indicated points are possible marked points for either $\text{Asc}(A) = (2n - c, c)$ or $\text{Asc}(A) = (c - 1, 2n - c + 1)$. The left picture shows that

$y_{(14,10)} = 5$ and $y_{(9,15)} = 4$, hence $x_{(14,10)} = 1$. The right picture shows that $y_{(17,7)} = 3 = y_{(6,18)}$, hence $x_{(17,7)} = 0$.

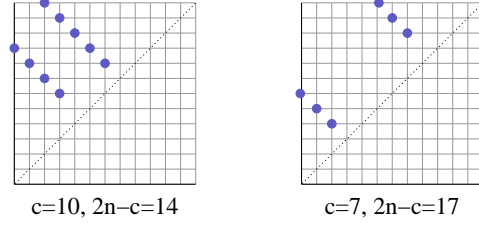


Figure 3: Possible marked points for $\text{Asc}(A) = (2n - c, c)$ or $\text{Asc}(A) = (c - 1, 2n - c + 1)$ when $n = 12$.

To complete the proof, we must show that $x_\lambda = 0$ whenever λ is a partition with at least three parts. By Fact 3, for each α with $\alpha_1 \geq \alpha_2$ and $\alpha_3 > 0$, it will suffice to define a bijection ϕ that sends a standard $(2), (n)$ -matrix A with $\text{Asc}(A) = (\alpha_1, \alpha_2, \alpha_3, \dots)$ to a standard $(2), (n)$ -matrix $\phi(A)$ with $\text{Asc}(\phi(A)) = (\alpha_2 - 1, \alpha_1 + 1, \alpha_3, \dots)$. Under the identification with lattice paths, each such A corresponds to a lattice path $p(A)$ starting with a north step. It follows from the marking rules that $m_1(A)$ is above the line $y = x$. However, $m_2(A)$ (and higher-indexed marked points) may lie on either side of $y = x$. Let $w(A)$ denote the first intersection of $p(A)$ with the line $y = x$ that occurs *after* $m_2(A)$. For the rest of the proof, it will be convenient to also consider an alternative path $p'(A)$ defined as follows. If $m_2(A)$ lies above $y = x$, let $p'(A) = p(A)$. If $m_2(A)$ lies below $y = x$, obtain $p'(A)$ from $p(A)$ by reflecting the portion of $p(A)$ before $w(A)$ about the line $y = x$. Let $m'_i(A)$ be the point on $p'(A)$ corresponding to the marked point $m_i(A)$ on $p(A)$. One checks that A (and hence $p(A)$) is still recoverable from $p'(A)$. Note that $p'(A)$ may start with an east step.

Fix $\alpha_1 \geq \alpha_2 > 0$ and a, b with $n \geq b > a \geq 0$ and $a + b = \alpha_1 + \alpha_2$. Define

$$Z^{(a,b)}(\alpha_1, \alpha_2) = \{A \in S_{2,n}((2), (n)) : \text{Asc}(A) = (\alpha_1, \alpha_2, \dots) \text{ and } m'_2(A) = (a, b)\}; \quad (13)$$

$$Z_{\circ}^{(a,b)}(\alpha_1, \alpha_2) = \{m'_1(A) : A \in Z^{(a,b)}(\alpha_1, \alpha_2)\}. \quad (14)$$

We claim that $|Z_{\circ}^{(a,b)}(\alpha_1, \alpha_2)| = \min(\alpha_2 - 1, a)$. To see this, first note that there are α_2 lattice points on the line $x + y = \alpha_1$ with x -coordinate at most a and y -coordinate strictly less than b . (We can never have $m'_1(A) = (a - \alpha_2, b)$ since $m'_2(A)$ lies above $y = x$.) If $a \geq \alpha_2 - 1$, then all of these points will have nonnegative x -coordinate. Otherwise, only $a + 1$ of them will. Suppose $b - \alpha_2 \geq a$. Then $m'_1(A)$ cannot be the point $(a, b - \alpha_2)$, but the other $\min(\alpha_2 - 1, a)$ points discussed will be possibilities for $m'_1(A)$. Suppose instead that $b - \alpha_2 < a$. Then $(a, b - \alpha_2)$ will be a possibility for $m'_1(A)$. But there will be a unique $i > 0$ for which $(a - i, b - \alpha_2 + i)$ is not, as it lies on one of the lines $y = x$ or $y = x + 1$. So the cardinality of $Z_{\circ}^{(a,b)}(\alpha_1, \alpha_2)$ is as claimed. Examples of the various cases above are illustrated in Figure 4. Denote the elements of $Z_{\circ}^{(a,b)}(\alpha_1, \alpha_2)$ by $P_1, \dots, P_{\min(\alpha_2 - 1, a)}$ in increasing order of y -coordinate.

We claim that $|Z_{\circ}^{(a,b)}(\alpha_2 - 1, \alpha_1 + 1)| = \min(\alpha_2 - 1, a)$ as well. By hypothesis, in this case there are α_2 points on the line $x + y = \alpha_2 - 1$ with nonnegative coordinates. We must restrict to those with x -coordinate at most a . There are $\min(\alpha_2, a + 1)$ such points. Then we must omit the unique point lying on either of the lines $y = x$ or $y = x + 1$. Denote the elements of $Z_{\circ}^{(a,b)}(\alpha_2 - 1, \alpha_1 + 1)$ by $R_1, \dots, R_{\min(\alpha_2 - 1, a)}$ in increasing order of x -coordinate.

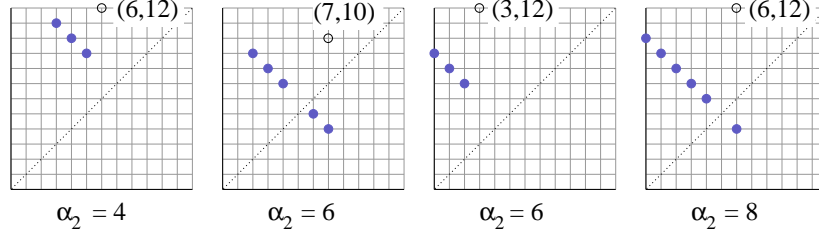


Figure 4: Possible marked points along the line $x + y = \alpha_1$ for various (a, b) and α_2 .

We can now define the bijection ϕ by assembling bijections that map each set $Z^{(a,b)}(\alpha_1, \alpha_2)$ onto $Z^{(a,b)}(\alpha_2 - 1, \alpha_1 + 1)$ for all a, b, α_1, α_2 as above. Given $A \in Z^{(a,b)}(\alpha_1, \alpha_2)$, write $m'(A) = (m'_1(A), m'_2(A), \dots, m'_k(A))$ where $k \geq 3$. Now $m'_1(A) = P_i$ for some $P_i \in Z^{(a,b)}(\alpha_1, \alpha_2)$. We define $\phi(A)$ by setting its marked points, $m'_i(\phi(A))$, to be equal to those of A except for the first one: $m'(\phi(A)) = (R_i, m'_2(A), m'_3(A), \dots)$. Since each path is determined by its marked points, this determines the image matrix $\phi(A)$. We know from above that the sets $\{P_i\}$ and $\{R_i\}$ are the same size. Furthermore, since ϕ replaces an element of $Z^{(a,b)}(\alpha_1, \alpha_2)$ with one of $Z^{(a,b)}(\alpha_2 - 1, \alpha_1 + 1)$ it induces a bijection between $Z^{(a,b)}(\alpha_1, \alpha_2)$ and $Z^{(a,b)}(\alpha_2 - 1, \alpha_1 + 1)$ as required. \square

Example 11. Consider the following element of $S_{(2),(16)}$:

$$A = \begin{bmatrix} 1 & 3 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 16 & 19 & 23 & 25 & 29 & 30 & 32 \\ 2 & 4 & 12 & 13 & 14 & 15 & 17 & 18 & 20 & 21 & 22 & 24 & 26 & 27 & 28 & 31 \end{bmatrix}.$$

One checks that $\text{Asc}(A) = (11, 5, 3, 6, 3, 3, 1)$. Using the notation from the proof of Theorem 10, $m_1(A) = m'_1(A) = P_4 = (2, 9)$, $m'_1(\phi(A)) = (4, 0)$ and $m_1(\phi(A)) = (0, 4)$. See Figure 5. So

$$\phi(A) = \begin{bmatrix} 1 & 2 & 3 & 4 & 10 & 12 & 17 & 18 & 20 & 21 & 22 & 23 & 25 & 29 & 30 & 32 \\ 5 & 6 & 7 & 8 & 9 & 11 & 13 & 14 & 15 & 16 & 19 & 24 & 26 & 27 & 28 & 31 \end{bmatrix},$$

and $\text{Asc}(\phi(A)) = (4, 12, 3, 6, 3, 3, 1)$ as required.

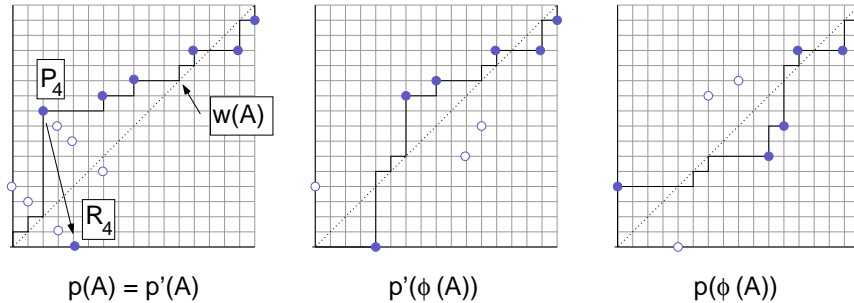


Figure 5: Relevant paths for Example 11. Points of interest that are not marked points for the indicated path are given by unfilled circles.

References

- [1] S. Assaf. A combinatorial proof of LLT and Macdonald positivity. 2010. Preprint.
- [2] J. O. Carbonara, J. B. Remmel, and M. Yang. A combinatorial rule for the Schur function expansion of the plethysm $s_{(1^a, b)}[p_k]$. *Linear and Multilinear Algebra*, 39(4):341–373, 1995.
- [3] Luisa Carini and J. B. Remmel. Formulas for the expansion of the plethysms $s_2[s_{(a, b)}]$ and $s_2[s_{(n^k)}]$. *Discrete Math.*, 193(1-3):147–177, 1998. Selected papers in honor of Adriano Garsia (Taormina, 1994).
- [4] C. Carré. Plethysm of elementary functions. *Bayreuth. Math. Schr.*, (31):1–18, 1990.
- [5] M. J. Carvalho and S. D’Agostino. Plethysms of Schur functions and the shell model. *J. Phys. A*, 34(7):1375–1392, 2001.
- [6] Y. M. Chen, A. M. Garsia, and J. Remmel. Algorithms for plethysm. In *Combinatorics and algebra (Boulder, Colo., 1983)*, volume 34 of *Contemp. Math.*, pages 109–153. Amer. Math. Soc., Providence, RI, 1984.
- [7] E. Egge, N. Loehr, and G. Warrington. From quasisymmetric expansions to Schur expansions via a modified inverse Kostka matrix. *European J. Combin.* To appear.
- [8] H. O. Foulkes. Concomitants of the quintic and sextic up to degree four in the coefficients of the ground form. *J. London Math. Soc.*, 25:205–209, 1950.
- [9] W. Fulton. *Young Tableaux; with applications to representation theory and geometry*, volume 35 of *London Mathematical Society Student Texts*. Cambridge University Press, New York, 1997.
- [10] A. M. Garsia and J. Haglund. A proof of the q, t -Catalan positivity conjecture. *Discrete Math.*, 256(3):677–717, 2002. LaCIM 2000 Conference on Combinatorics, Computer Science and Applications (Montreal, QC).
- [11] Ira M. Gessel. Multipartite P -partitions and inner products of skew Schur functions. In *Combinatorics and algebra (Boulder, Colo., 1983)*, volume 34 of *Contemp. Math.*, pages 289–317. Amer. Math. Soc., Providence, RI, 1984.
- [12] J. Haglund. A combinatorial model for the Macdonald polynomials. *Proc. Natl. Acad. Sci. USA*, 101(46):16127–16131 (electronic), 2004.
- [13] J. Haglund, M. Haiman, and N. Loehr. A combinatorial formula for Macdonald polynomials. *J. Amer. Math. Soc.*, 102:2690–2696, 2005.
- [14] T. M. Langley and J. B. Remmel. The plethysm $s_\lambda[s_\mu]$ at hook and near-hook shapes. *Electron. J. Combin.*, 11(1):Research Paper 11, 26 pp. (electronic), 2004.
- [15] Alain Lascoux, Bernard Leclerc, and Jean-Yves Thibon. Ribbon tableaux, Hall-Littlewood functions and unipotent varieties. *Sém. Lothar. Combin.*, 34:Art. B34g, approx. 23 pp. (electronic), 1995.

- [16] D. E. Littlewood. Invariant theory, tensors and group characters. *Philos. Trans. Roy. Soc. London. Ser. A.*, 239:305–365, 1944.
- [17] N. Loehr and J. Remmel. A computational and combinatorial exposé of plethystic calculus. *J. Algebraic Combin.* To appear.
- [18] Nicholas A. Loehr and Gregory S. Warrington. Nested quantum Dyck paths and $\nabla(s_\lambda)$. *Int. Math. Res. Not. IMRN*, (5):Art. ID rnm 157, 29, 2008.
- [19] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford Mathematical Monographs. Oxford University Press, New York, second edition, 1995. With contributions by A. Zelevinsky, Oxford Science Publications.
- [20] Claudia Malvenuto and Christophe Reutenauer. Plethysm and conjugation of quasi-symmetric functions. *Discrete Math.*, 193(1-3):225–233, 1998. Selected papers in honor of Adriano Garsia (Taormina, 1994).
- [21] S. P. O. Plunkett. On the plethysm of S -functions. *Canad. J. Math.*, 24:541–552, 1972.
- [22] R. M. Thrall. On symmetrized Kronecker powers and the structure of the free Lie ring. *Amer. J. Math.*, 64:371–388, 1942.
- [23] Brian G. Wybourne. *Symmetry principles and atomic spectroscopy*. Wiley-Interscience [A division of John Wiley & Sons], New York-London-Sydney, 1970. Including an appendix of tables by P. H. Butler.
- [24] Mei Yang. An algorithm for computing plethysm coefficients. In *Proceedings of the 7th Conference on Formal Power Series and Algebraic Combinatorics (Noisy-le-Grand, 1995)*, volume 180, pages 391–402, 1998.
- [25] Mei Yang. The first term in the expansion of plethysm of Schur functions. *Discrete Math.*, 246(1-3):331–341, 2002. Formal power series and algebraic combinatorics (Barcelona, 1999).