Square q, t-lattice paths and $\nabla(p_n)$

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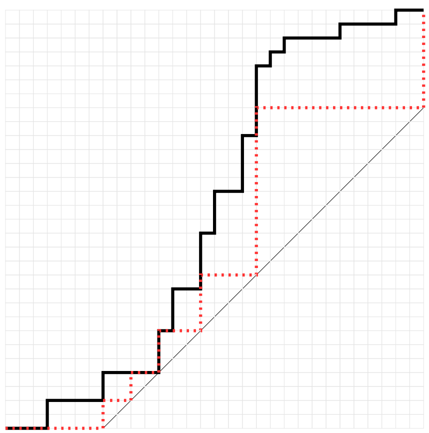
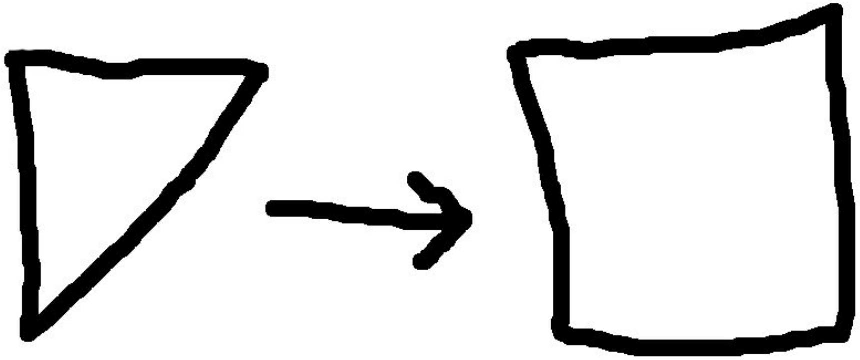
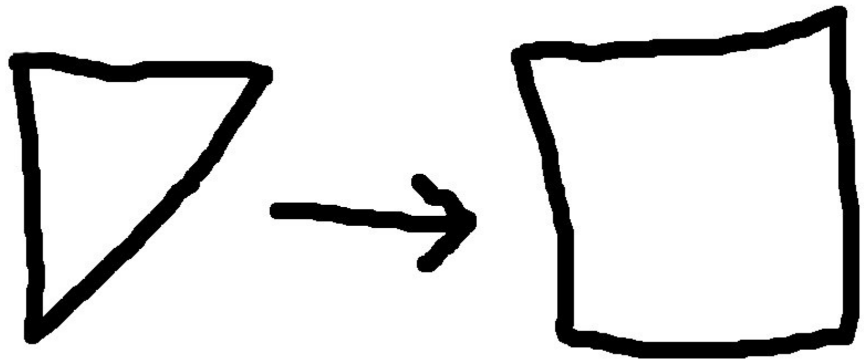


FPSAC — 20 June, 2005

Overview



Overview



- We generalize the $\mathbf{area}$, $\mathbf{boun}$, and $\mathbf{dinv}$ statistics.
- We conjecture a combinatorial formula for $\boldsymbol{\nabla}(\boldsymbol{p_n})$.

Situation circa 1996

Given (Garsia-Haiman): Rational functions $OC_n(q,t)$

satisfying $OC_n(q,t) = OC_n(t,q),$

$$OC_n(1,1) = \frac{1}{n+1}\binom{2n}{n} = C_n,$$

$$q^{\binom{n}{2}} OC_n(q,1/q) = \frac{1}{[n+1]_q}\begin{bmatrix} 2n \\ n,n \end{bmatrix}_q,$$

$$OC_n(1,q) = OC_n(q,1) = \sum_{D \in \mathcal{D}_n} q^{\mathrm{area}(D)}.$$

Wanted: $OC_n(q,t) = \sum_{D \in \mathcal{D}_n} q^{\mathrm{area}(D)} t^{\mathrm{tstat}(D)}.$

Situation circa 2000

$\mathbf{boun}$ (Haglund) and $\mathbf{dinv}$ (Haiman) are proposed.
Note: There exists a bijection $\phi : \mathcal{D}_n \to \mathcal{D}_n$ taking

$$(\mathrm{dinv}(D), \mathrm{area}(D)) \mapsto (\mathrm{area}(\phi(D)), \mathrm{boun}(\phi(D))).$$

Define

$$C_n(q,t) = \sum_{D \in \mathcal{D}_n} q^{\mathrm{area}(D)} t^{\mathrm{boun}(D)} = \sum_{D \in \mathcal{D}_n} q^{\mathrm{dinv}(D)} t^{\mathrm{area}(D)}.$$

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Theorem (Garsia-Haglund).

$$C_n(q,t) = OC_n(q,t).$$

∇

Let $\mu \vdash n$ and $n(\mu) = \sum_i (i-1)\mu_i$.
The nabla operator (F. Bergeron-Garsia) is the unique F-linear map on Λ_F^n defined by:

$$\nabla(\tilde{H}_\mu) = q^{n(\mu')} t^{n(\mu)} \tilde{H}_\mu,$$

where $\{\tilde{H}_\mu\}$ are the modified Macdonald polynomials.

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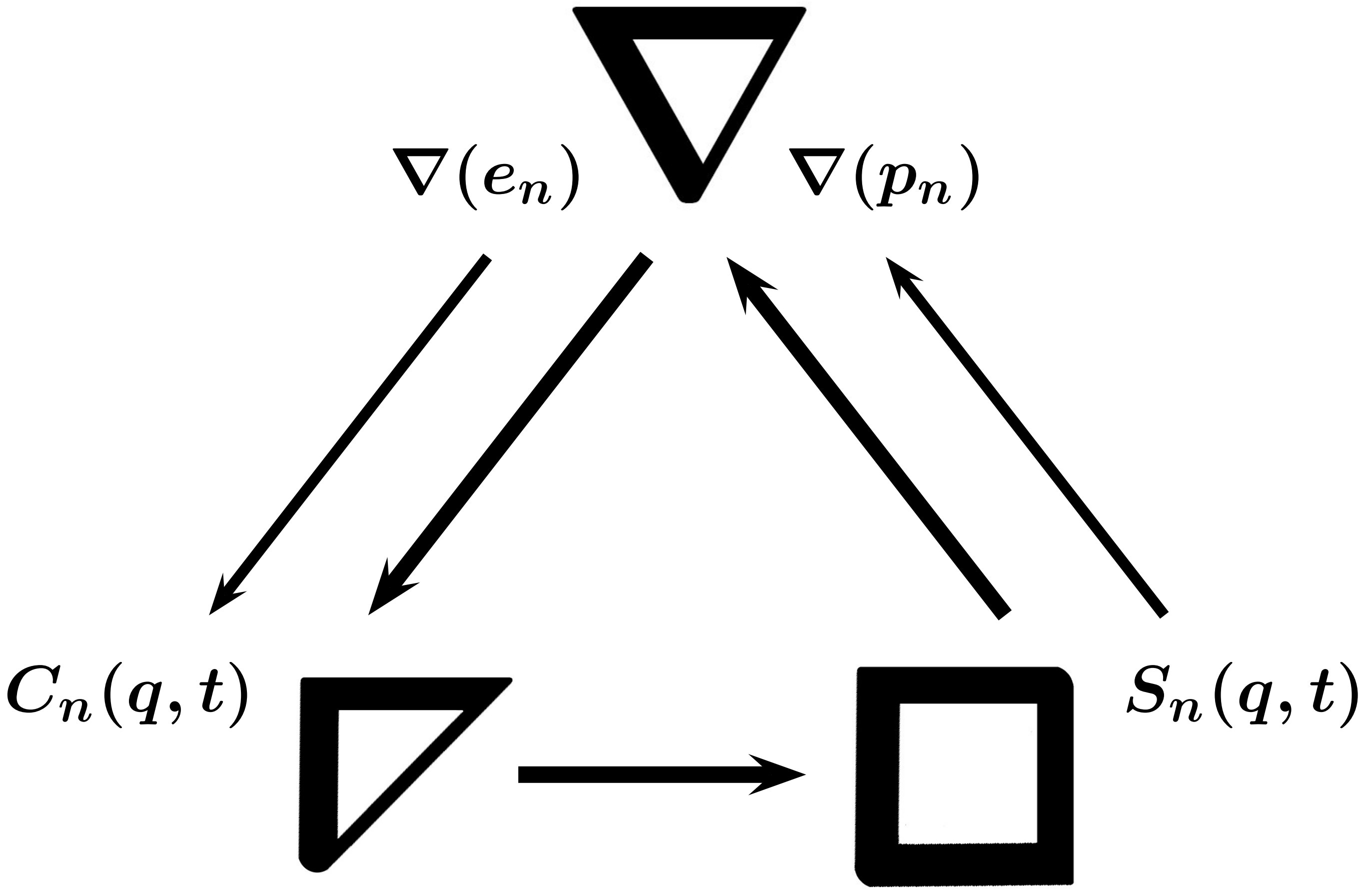
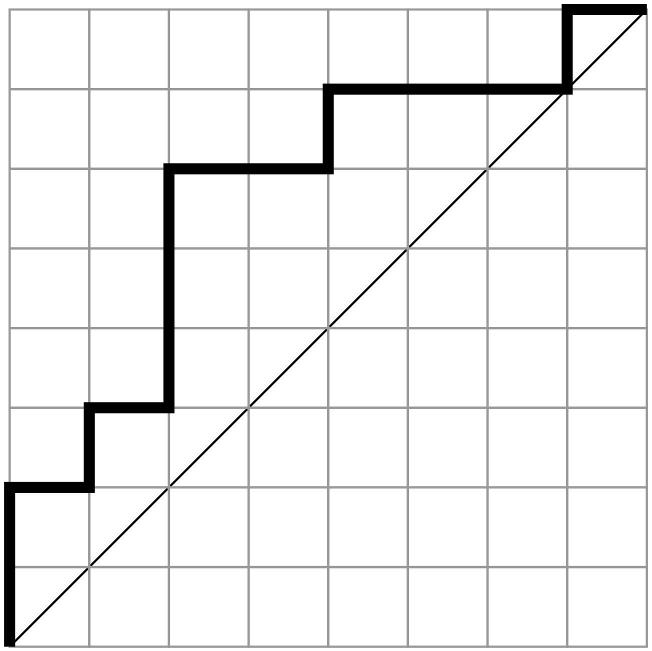
Theorem (Haiman). $\langle \nabla(e_n), s_{1^n} \rangle = OC_n(q,t)$.

Big picture

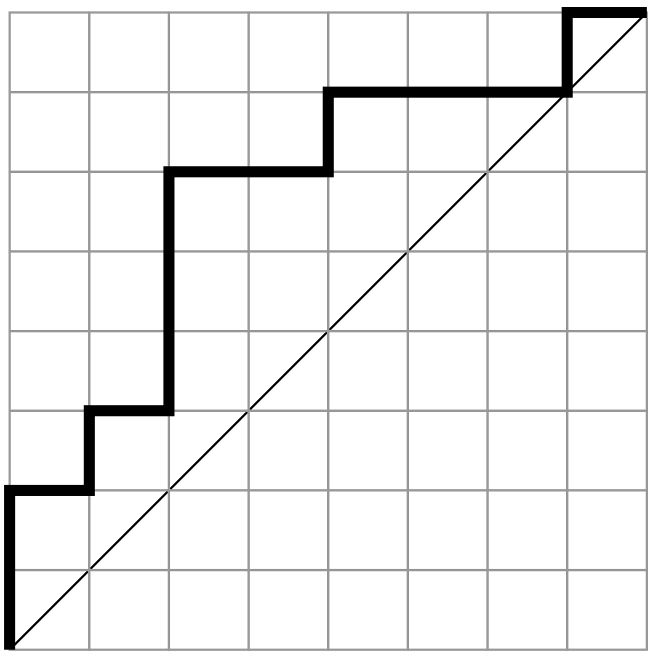


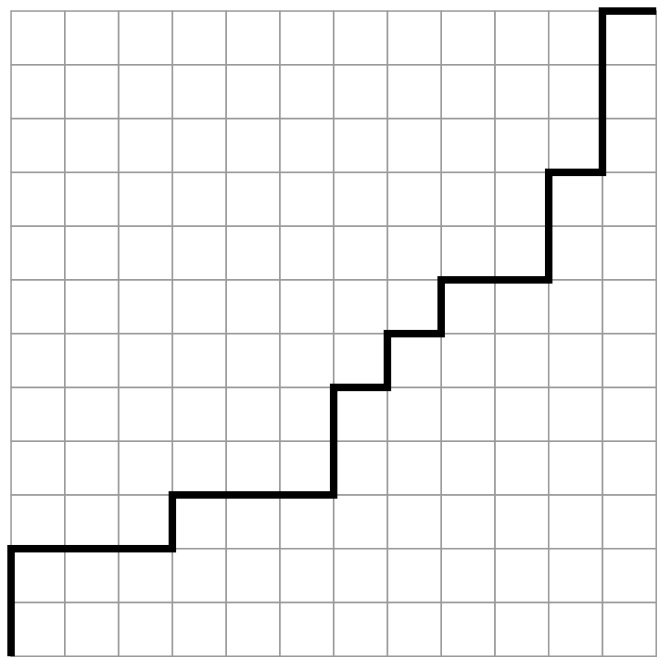
area, boun, and dinv



$$\#\mathcal{D}_n = \frac{1}{1+n}\binom{2n}{n}$$

area, boun, and dinv

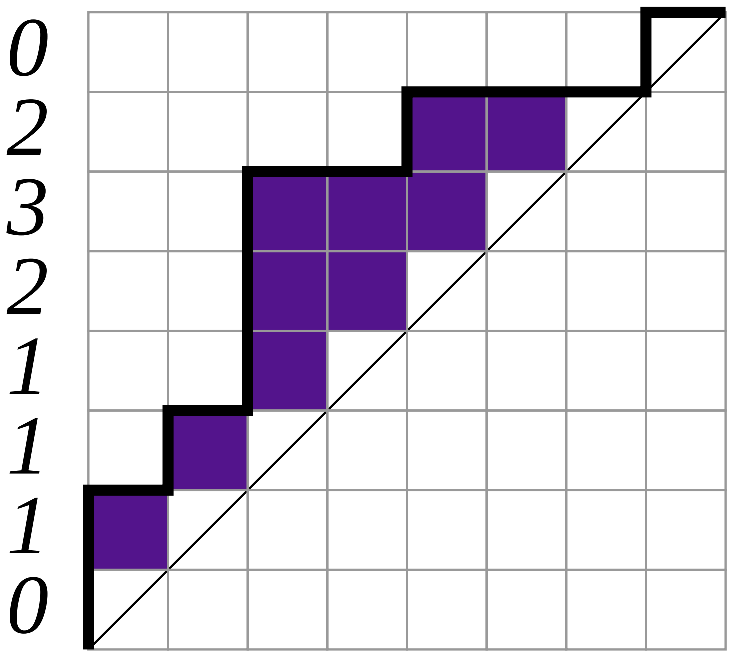


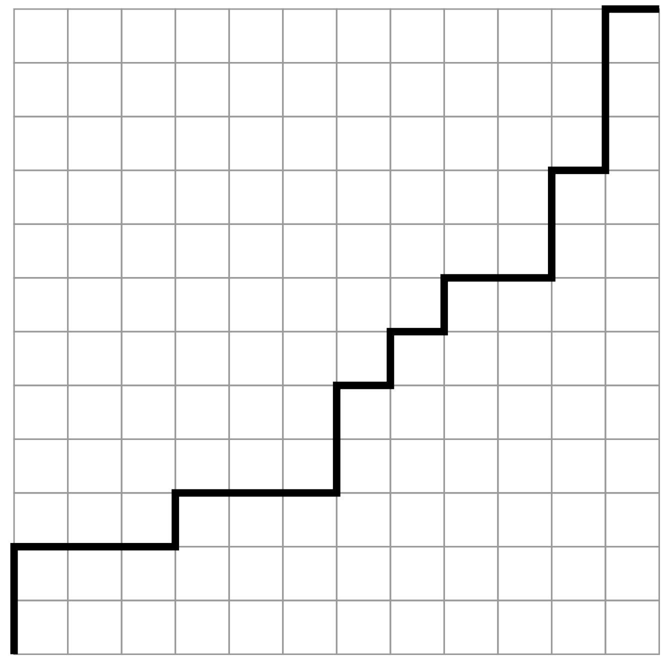


$$\#\mathcal{D}_n = \frac{1}{1+n}\binom{2n}{n}$$

$$\#\mathcal{SQ}_n = \binom{2n}{n}$$

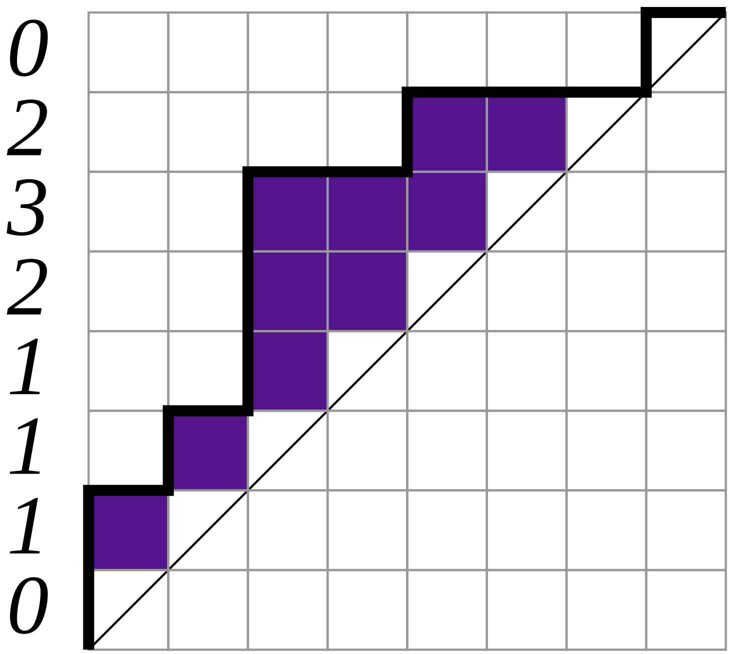
area, boun, and dinv

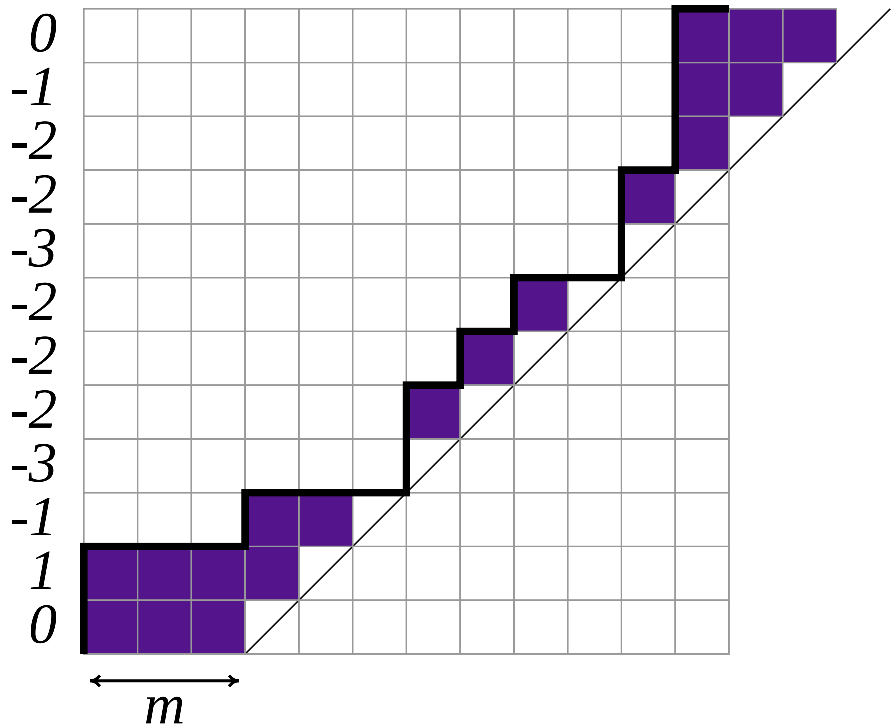




$$\mathbf{area}(\boldsymbol{D}) = \sum_{i=0}^{n-1} g_i(\boldsymbol{D})$$

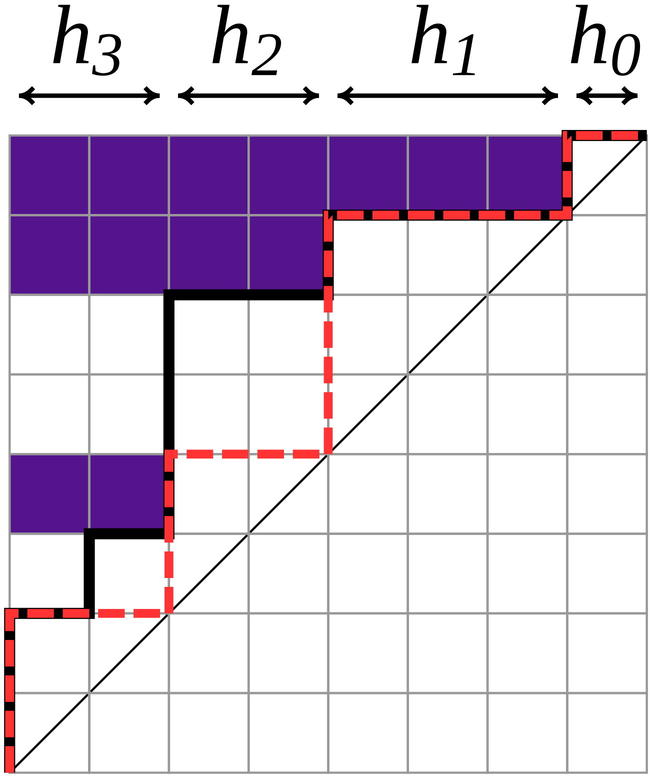
area, boun, and dinv

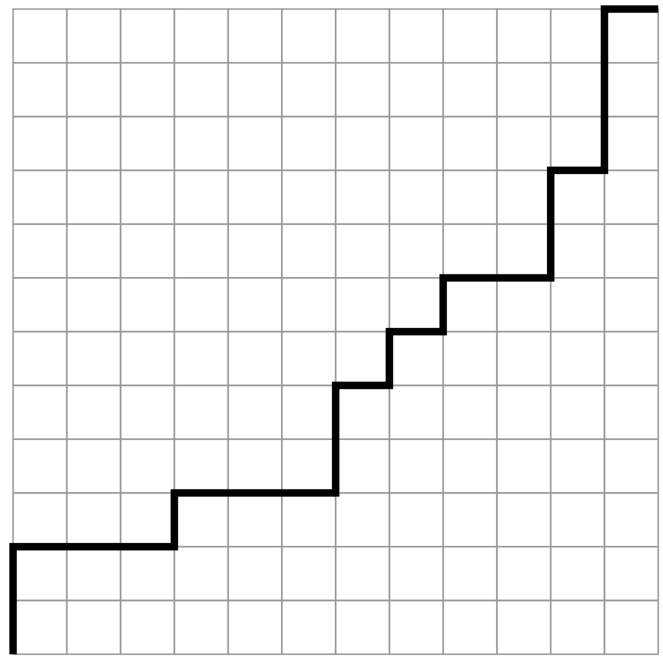




$$\mathbf{area}(S) = \sum_{i=0}^{n-1} (m + g_i(S)) + \binom{m}{2}$$

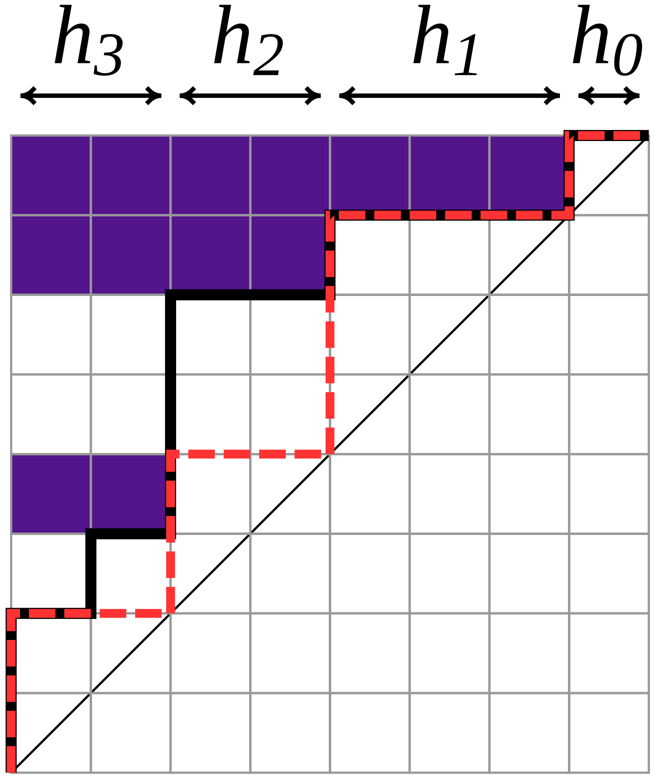
area, boun, and dinv

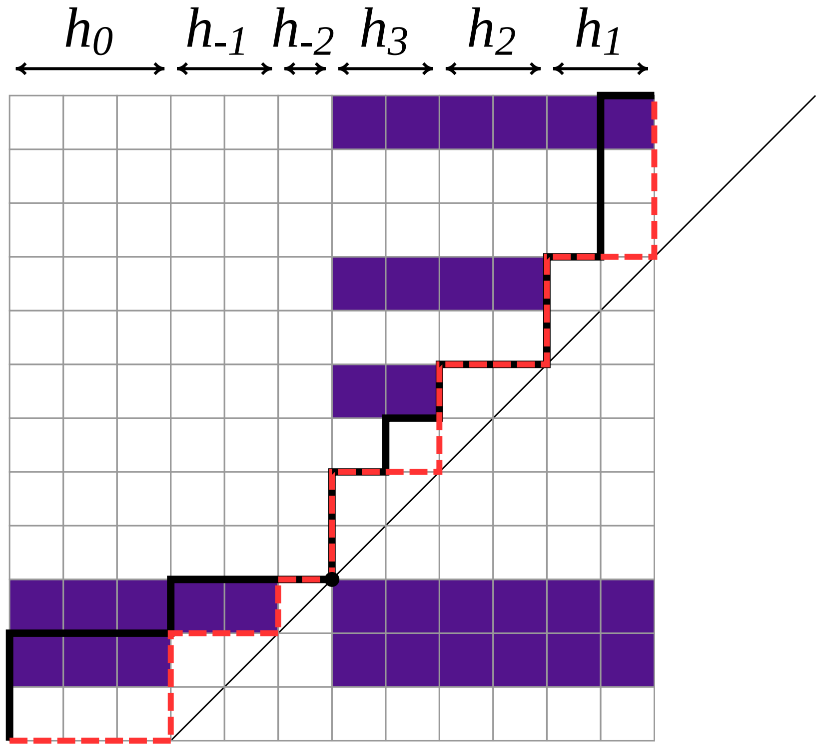




$$\mathbf{boun}(D) = \sum_{i=0} i h_i(D)$$

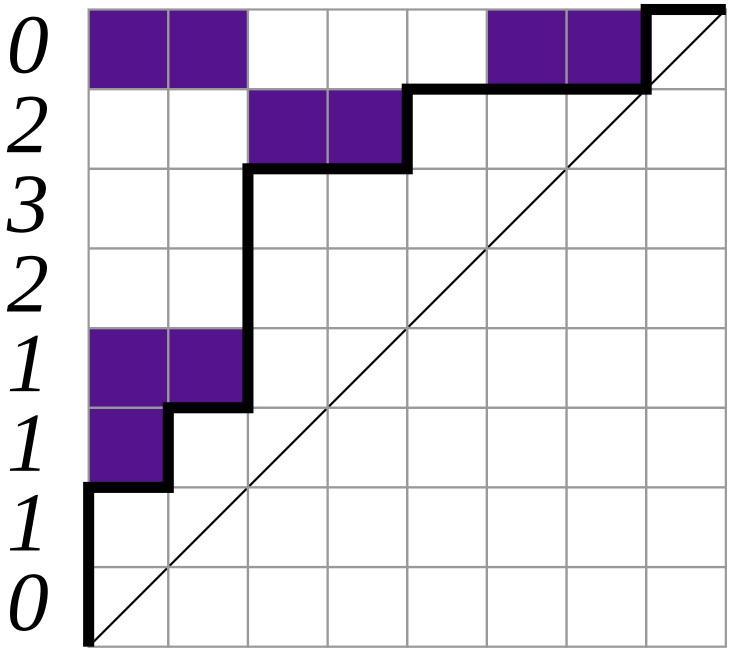
area, boun, and dinv

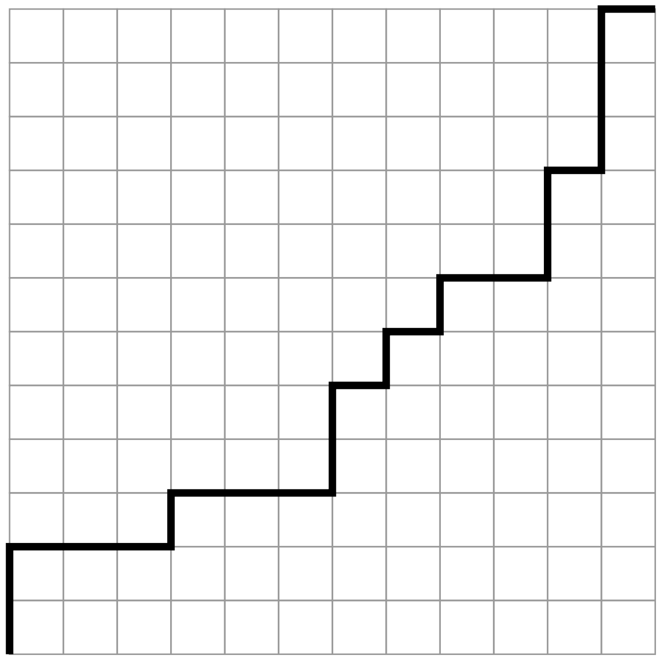




$$\mathbf{boun}(S) = \sum_{i=0} i h_{i+u}(S)$$

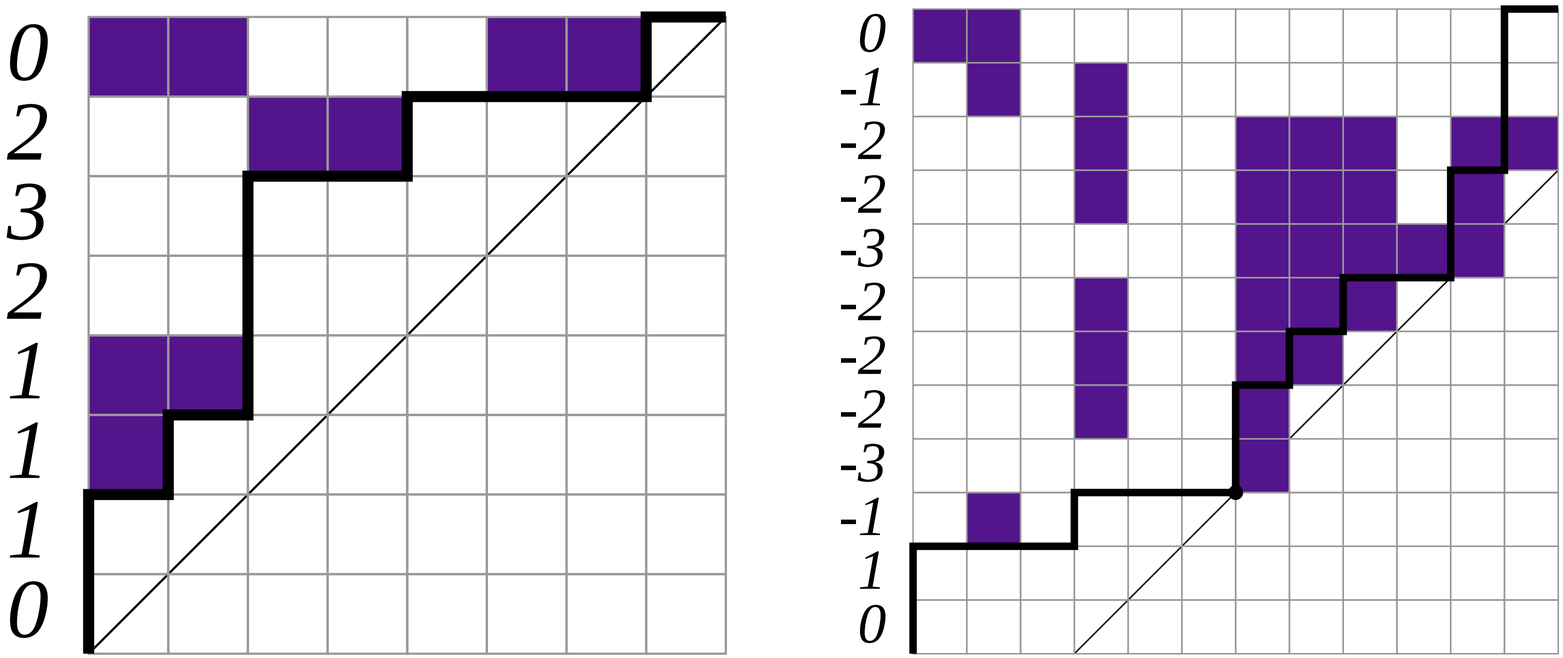
area, boun, and dinv





$$\mathbf{dinv}(D) = \sum_{i<j} \chi[g_i(D) - g_j(D) \in \{0, 1\}]$$

area, boun, and dinv



$$\mathbf{dinv}(S) = \sum_{i<j} \chi[g_i(S) - g_j(S) \in \{0, 1\}]$$

$$+ \sum_{i} \chi[g_i(S) < -1]$$

The q, t-Square numbers

There exists a bijection $\phi : \mathcal{SQ}_n \to \mathcal{SQ}_n$ taking

$$(\mathrm{dinv}(S), \mathrm{area}(S)) \mapsto (\mathrm{area}(\phi(S)), \mathrm{boun}(\phi(S))).$$

Define

$$S_n(q,t) = \sum_{S \in \mathcal{SQ}_n} q^{\mathrm{area}(S)} t^{\mathrm{boun}(S)} = \sum_{S \in \mathcal{SQ}_n} q^{\mathrm{dinv}(S)} t^{\mathrm{area}(S)}.$$

The q, t-Square numbers

There exists a bijection $\phi : \mathcal{SQ}_n \longrightarrow \mathcal{SQ}_n$ taking

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Conjecture. For all $n \geq 1$,

$$S_n(q,t) = S_n(t,q).$$

$t = 1/q$

Theorem (Haglund).

$$q^{\binom{n}{2}} C_n(q, 1/q) = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n, n \end{bmatrix}_q .$$

$t = 1/q$

Theorem (Haglund).

$$q^{\binom{n}{2}} C_n(q, 1/q) = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n, n \end{bmatrix}_q.$$

Theorem. For all $n \geq 1$,

$$q^{\binom{n}{2}} S_n(q, 1/q) = \frac{2}{1+q^n} \begin{bmatrix} 2n \\ n, n \end{bmatrix}_q = 2 \begin{bmatrix} 2n-1 \\ n, n-1 \end{bmatrix}_q.$$

Accounting for the 2

Define

$$
\begin{aligned}
{}^{N}S_n(q,t) &= \sum_{S \in {}^{N}\mathcal{SQ}_n} q^{\mathrm{dinv}(S)} t^{\mathrm{area}(S)}, \\
{}^{E}S_n(q,t) &= \sum_{S \in {}^{E}\mathcal{SQ}_n} q^{\mathrm{dinv}(S)} t^{\mathrm{area}(S)}.
\end{aligned}
$$

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Define

$$
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\end{aligned}
$$

Theorem.

1. There exists a bijection ${}^{N}\mathcal{SQ}_n \to {}^{E}\mathcal{SQ}_n$ that preserves dinv and area.
2. ${}^{N}S_n(q,t) = {}^{E}S_n(q,t) = S_n(q,t)/2$.

∇ and ${}^{E}\mathcal{SQ}_n$

Theorem (Can-Loehr-Haglund). For all $n \geq 1$,

$$(-1)^{n-1}\langle \nabla(p_n), s_{1^n}\rangle = \sum_{S \in {}^{E}\mathcal{SQ}_n} q^{\mathrm{dinv}(S)} t^{\mathrm{area}(S)}.$$

Labeled versions

Fix n and N with $n \leq N \leq \infty$.
Let $r = r_0 \ldots r_{n-1}$ with $r_i \in \{1, 2, \ldots, N\}$.
Let $\mathcal{SQF}_n$ denote the set of all pairs (S, r) such that:

1. S is a path in ${}^E\mathcal{SQ}_n$ and
2. $r = r_0 \ldots r_{n-1}$ with $r_i \in \{1, 2, \ldots, N\}$ such that $g_{i+1}(S) = g_i(S) + 1$ implies $r_i < r_{i+1}$.

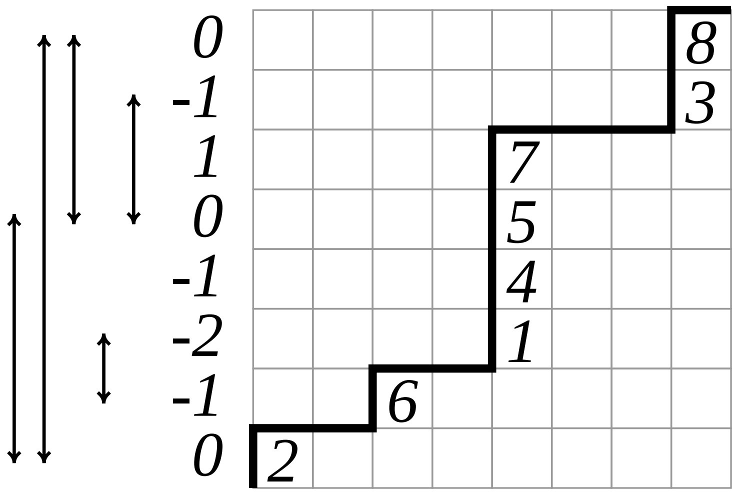
Let $\mathcal{SQH}_n \subset \mathcal{SQF}_n$ be given by

$$\mathcal{SQH}_n = \{(S, r) : \{r_0, \ldots, r_{n-1}\} = \{1, 2, \ldots, n\}\}.$$

Labeled versions

Given $(S, r) \in \mathcal{SQF}_n$, define $\mathbf{area}(S, r) = \mathbf{area}(S)$ and

$$\mathbf{dinv}(S, r) = \sum_{i<j} \chi[(g_i(S) - g_j(S) = 0 \text{ and } r_i < r_j) \text{ or } (g_i(S) - g_j(S) = 1 \text{ and } r_i > r_j)] + \sum_{i=0}^{n-1} \chi[g_i(S) < -1].$$



∇ conjectures

"Hilbert series" conjecture. For all $n \geq 1$,

$$(-1)^{n-1}\langle \nabla(p_n), h_{1^n} \rangle = \sum_{(S,r)\in \mathcal{SQH}_n} q^{\mathrm{dinv}(S,r)} t^{\mathrm{area}(S,r)}.$$

"Frobenius series" conjecture. For all $n \geq 1$,

$$(-1)^{n-1}\nabla(p_n[\vec{z}]) = \sum_{(S,r)\in \mathcal{SQF}_n} q^{\mathrm{dinv}(S,r)} t^{\mathrm{area}(S,r)} \prod_{i=0}^{n-1} z_{r_i}.$$