SQUARE q,t-LATTICE PATHS AND $\nabla(p_n)$

NICHOLAS A. LOEHR AND GREGORY S. WARRINGTON

ABSTRACT. The *combinatorial q,t-Catalan numbers* are weighted sums of Dyck paths introduced by J. Haglund and studied extensively by Haglund, Haiman, Garsia, Loehr, and others. The q,t-Catalan numbers, besides having many subtle combinatorial properties, are intimately connected to symmetric functions, algebraic geometry, and Macdonald polynomials. In particular, the n'th q,t-Catalan number is the Hilbert series for the module of diagonal harmonic alternants in $2n$ variables; it is also the coefficient of s_{1^n} in the Schur expansion of $\nabla(e_n)$. Using q,t-analogues of labelled Dyck paths, Haglund et al. have proposed combinatorial conjectures for the monomial expansion of $\nabla(e_n)$ and the Hilbert series of the diagonal harmonics modules.

This article extends the combinatorial constructions of Haglund et al. to the case of lattice paths contained in squares. We define and study several q,t-analogues of these lattice paths, proving combinatorial facts that closely parallel corresponding results for the q,t-Catalan polynomials. We also conjecture an interpretation of our combinatorial polynomials in terms of the nabla operator. In particular, we conjecture combinatorial formulas for the monomial expansion of $\nabla(p_n)$, the "Hilbert series" $\langle \nabla(p_n), h_{1^n} \rangle$, and the sign character $\langle \nabla(p_n), s_{1^n} \rangle$.

1. Introduction

In 1996, A. Garsia and M. Haiman introduced a two-variable analogue of the Catalan numbers called the *q,t-Catalan numbers* [7]. Garsia and Haiman's definition of the q,t-Catalan, which arose from their study of Macdonald polynomials and diagonal harmonics, was quite complicated. Several years later, J. Haglund [8] conjectured an elementary combinatorial definition of the q,t-Catalan numbers as weighted sums of Dyck paths relative to two statistics called area and bounce. Shortly thereafter, Haiman proposed an equivalent combinatorial interpretation involving area and a third statistic called dinv. Garsia and Haglund eventually proved that the two combinatorial definitions were equivalent to the original definition of Garsia and Haiman [5, 6]. Haiman proved many of the conjectures relating the q,t-Catalan numbers to the representation theory of diagonal harmonics modules and the algebraic geometry of the Hilbert scheme

[17, 18]. Meanwhile, various authors studied the subtle combinatorial properties of the combinatorial q, t-Catalan numbers and their generalizations [4, 9, 13, 14, 19, 20, 21, 22, 23, 24]. Surveys of different aspects of this research can be found in [15, 16, 19], and especially [11].

This article discusses a generalization of the combinatorial q, t-Catalan numbers in which Dyck paths are replaced by lattice paths inside squares. We develop the combinatorial theory of these "square q, t-lattice paths," which closely parallels the corresponding theory for the q, t-Catalan numbers. We also conjecture algebraic interpretations for our combinatorial generating functions in terms of the nabla operator introduced by F. Bergeron and Garsia [1, 2, 3]. In particular, we conjecture a combinatorial formula for the monomial expansion of $\nabla(p_n)$ that is quite similar to a formula for $\nabla(e_n)$ conjectured in [13].

To motivate and organize our work on lattice paths inside squares, we begin by quickly reviewing the combinatorial and algebraic results associated with the combinatorial q, t-Catalan numbers. The main body of the paper discusses the corresponding results and conjectures for our square q, t-lattice paths.

1.1. Combinatorial Aspects of the q, t-Catalan Numbers.
This section reviews the essential definitions and combinatorial results involving the q, t-Catalan numbers.

1. Lattice Paths and Dyck Paths. A *lattice path* in a $c \times d$ rectangle is a path from $(0, 0)$ to (c, d) consisting of c east steps and d north steps of length 1. Such a path can be represented as a word $w = w_1 \cdots w_{c+d}$ with d zeroes (encoding north steps) and c ones (encoding east steps). Let $\mathcal{R}_{c,d}$ be the set of lattice paths from $(0, 0)$ to (c, d). A *Dyck path of order n* is a lattice path in an $n \times n$ rectangle that never visits any point (x, y) with $y < x$. Let $\mathcal{D}_n$ be the set of Dyck paths of order n.

2. Classical Statistics on Paths. For any logical statement A, write $\chi(A) = 1$ if A is true, and $\chi(A) = 0$ if A is false. Let $P \in \mathcal{R}_{c,d}$ be encoded by the word $w = w_1 w_2 \cdots w_{c+d}$. Define the *major index*, *inversions*, and *lower area* of P by the formulas $\mathrm{maj}(P) = \sum_{i<c+d} i\chi(w_i > w_{i+1})$, $\mathrm{inv}(P) = \sum_{i<j} \chi(w_i > w_j)$, and $\mathrm{ar}(P) = cd - \mathrm{inv}(P)$. The *$q$-binomial coefficient* is given by any of the following formulas:

$$(1) \qquad \begin{bmatrix} c + d \\ c, d \end{bmatrix}_q = \sum_{P \in \mathcal{R}_{c,d}} q^{\mathrm{ar}(P)} = \sum_{P \in \mathcal{R}_{c,d}} q^{\mathrm{inv}(P)} = \sum_{P \in \mathcal{R}_{c,d}} q^{\mathrm{maj}(P)}.$$

For $D \in \mathcal{D}_n \subset \mathcal{R}_{n,n}$, let $g_i(D)$ be the number of complete lattice squares between D and the line $y = x$ in the i'th row from the bottom ($0 \le i < n$). Define the *area vector* of D to be $g(D) = (g_0(D), \ldots, g_{n-1}(D))$. Set $\mathrm{area}(D) = \sum_{i=0}^{n-1} g_i(D) = \mathrm{ar}(D) - \binom{n+1}{2}$.

3. The Statistics dinv *and* bounce. Given $D \in \mathcal{D}_n$, Haiman defined the *d-inversions of* D by setting

$$\text{dinv}(D) = \sum_{i<j} \chi(g_i(D) - g_j(D) \in \{0,1\}).$$

Haglund defined a *bounce path* and *bounce statistic* for D as follows. A ball starts at (n,n) and makes a succession of alternating horizontal and vertical moves $H_0(D), V_0(D), H_1(D), V_1(D), \ldots, H_s(D), V_s(D)$ leading west and south to $(0,0)$. The moves $H_i(D)$ and $V_i(D)$ will consist of $h_i(D)$ west steps and $v_i(D) = h_i(D)$ south steps, respectively. After the first i pairs of moves, the ball has reached the point (n_i, n_i) on the main diagonal, where $n_i = n - \sum_{j<i} h_j(D)$. At the next step, the ball moves west $h_i(D)$ units until it is "blocked" by the north step of D *ending* at $(n_i - h_i(D), n_i)$. The ball then moves south $v_i(D) = h_i(D)$ units to return to the main diagonal. The bouncing stops when the ball reaches $(0,0)$. The lattice path traced out by the ball is called the *bounce path* bpath(D). Note that $\sum_{i \geq 0} v_i(D) = \sum_{i \geq 0} h_i(D) = n$. Call the north steps that block the ball *blocking north steps*. Define bounce$(D) = \sum_{i \geq 0} iv_i(D) = \sum_{i \geq 0} ih_i(D)$, which is the number of lattice squares in the rows west of the blocking north steps. See Figure 1 for an example; the cells contributing to bounce(D) are marked by X's, and the blocking north steps are cross-hatched. We have maj$(D) = 34$, inv$(D) = 18$, area$(D) = 10$, dinv$(D) = 10$, and bounce$(D) = 14$.
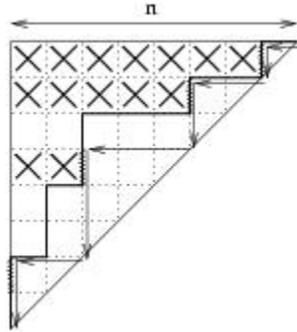

FIGURE 1. Bouncing through a Dyck path.

4. Combinatorial q,t-Catalan Numbers. We define the combinatorial q,t-Catalan numbers by the equivalent formulas

$$C_n(q,t) = \sum_{D \in \mathcal{D}_n} q^{\text{dinv}(D)} t^{\text{area}(D)} = \sum_{D \in \mathcal{D}_n} q^{\text{area}(D)} t^{\text{bounce}(D)}.$$

The equivalence of these definitions is proved by exhibiting a bijection $\phi : \mathcal{D}_n \to \mathcal{D}_n$ that maps the ordered pair of statistics (dinv, area) to

(area, bounce). We describe a more general bijection in §2.2, which includes ϕ as a special case.

5. Symmetry Properties. We have the *univariate symmetry* $C_n(q, 1) = C_n(1, q)$, which says that the statistics area, dinv, and bounce are equidistributed on $\mathcal{D}_n$. In fact, we even have the *joint symmetry* $C_n(q, t) = C_n(t, q)$, which follows from Garsia and Haglund's long proof linking $C_n(q, t)$ to the nabla operator [5, 6]. The map ϕ provides a bijective proof of univariate symmetry; giving a bijective proof of joint symmetry is an open problem.

6. Recursion. For $0 \leq k \leq n$, let $\mathcal{D}_{n,k}$ consist of all Dyck paths $D \in \mathcal{D}_n$ ending with exactly k east steps; $D \in \mathcal{D}_{n,k}$ iff $h_0(D) = k$. Set

$$(2) \qquad C_{n,k}(q, t) = \sum_{D \in \mathcal{D}_{n,k}} q^{\mathrm{area}(D)} t^{\mathrm{bounce}(D)}.$$

Haglund proved the recursion

$$(3) \quad C_{n,k}(q, t) = q^{k(k-1)/2} t^{n-k} \sum_{r=0}^{n-k} \begin{bmatrix} r + k - 1 \\ r, k - 1 \end{bmatrix}_q C_{n-k,r}(q, t) \text{ for } 1 \leq k \leq n$$

with initial conditions $C_{n,0}(q, t) = \chi(n = 0)$ for all $n \geq 0$ [8]. Since $C_n(q, t) = t^{-n} C_{n+1,1}(q, t)$, this recursion uniquely determines the q, t-Catalan numbers.

7. Fermionic Formula. Haglund also derived an explicit "fermionic" formula for $C_n(q, t)$ [8]:

$$(4) \qquad C_n(q, t) = \sum_{\substack{w_0 + \cdots + w_s = n \\ w_i > 0}} q^{\sum_i \binom{w_i}{2}} t^{\sum_i i w_i} \prod_{i=0}^{s-1} \begin{bmatrix} w_{i+1} + w_i - 1 \\ w_{i+1}, w_i - 1 \end{bmatrix}_q.$$

8. Specialization at $t = 1/q$. Haglund [8] and Loehr [24] gave algebraic and bijective proofs (respectively) of the respective formulas

$$(5) \qquad q^{\binom{n}{2}} C_{n,k}(q, 1/q) = \frac{[k]_q}{[n]_q} \begin{bmatrix} 2n - k - 1 \\ n - k, n - 1 \end{bmatrix}_q q^{(k-1)n};$$

$$(6) \quad q^{\binom{n+1}{2} - nk} C_{n,k}(q, 1/q) = \begin{bmatrix} 2n - k - 1 \\ n - k, n - 1 \end{bmatrix}_q - q^k \begin{bmatrix} 2n - k - 1 \\ n - k - 1, n \end{bmatrix}_q.$$

It follows that $q^{\binom{n}{2}} C_n(q, 1/q)$ is given by the formulas

$$(7) \qquad \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n, n \end{bmatrix}_q = \begin{bmatrix} 2n \\ n, n \end{bmatrix}_q - q \begin{bmatrix} 2n \\ n - 1, n + 1 \end{bmatrix}_q = \sum_{D \in \mathcal{D}_n} q^{\mathrm{maj}(D)}.$$

9. Statistics for Labelled Paths. The area and dinv statistics for Dyck paths extend naturally to *labelled Dyck paths*. A *labelled Dyck path* of order n is a path $D \in \mathcal{D}_n$ in which each vertical step is assigned a label between 1 and n. We require that the labels of vertical steps in the same column strictly increase from bottom to top. Let $\mathcal{P}_n$ denote the set of all such objects with distinct labels; let $\mathcal{Q}_n$ denote the set of all such objects where labels

may be repeated (subject to the increasing-column condition). An object $Q \in \mathcal{Q}_n$ is uniquely determined by the pair $(g(Q), r(Q))$, where $g(Q) = (g_0(Q), \ldots, g_{n-1}(Q))$ is the area vector of the path Q (ignoring labels), and $r(Q) = (r_0(Q), \ldots, r_{n-1}(Q))$ is the sequence of labels in Q from bottom to top. We call $r(Q)$ the *label vector* of Q. Define $c_Q(j)$ to be the number of j's in the label vector $r(Q)$. Define $\text{area}(Q) = \sum_{i=0}^{n-1} g_i(Q)$ and

$$\text{dinv}(Q) = \sum_{i<j} \chi((g_i = g_j \text{ and } r_i < r_j) \text{ or } (g_i = g_j + 1 \text{ and } r_i > r_j)).$$

Finally, define generating functions

$$H_n(q,t) = \sum_{P \in \mathcal{P}_n} q^{\text{area}(P)} t^{\text{dinv}(P)}, \quad F_n(\vec{z}; q, t) = \sum_{Q \in \mathcal{Q}_n} q^{\text{area}(P)} t^{\text{dinv}(P)} \prod_{j \geq 1} z_j^{c_Q(j)}.$$

Many combinatorial properties of the q,t-Catalan numbers extend to these polynomials [13, 14, 21], including: univariate symmetry in q and t, conjectural joint symmetry, definitions via bounce-like statistics, fermionic formulas, recursions, and specializations at $t = 1/q$.

1.2. Algebraic Aspects of the q,t-Catalan Numbers. This section states the principal theorems and conjectures connecting Haglund's combinatorial q,t-Catalan numbers (and their extensions H_n and F_n) to the theory of Macdonald polynomials, diagonal harmonics modules, and symmetric functions. We first review some standard definitions and results in symmetric function theory and representation theory; see [25, 26] for more details.

1. Partitions. We use standard notation for integer partitions, as in [25]. In particular, if $\mu \vdash n$ and c is a cell in the diagram of μ, we use the notation μ', $\leq$, $a(c)$, $a'(c)$, $l(c)$, $l'(c)$, and $n(\mu)$ to denote (respectively) the transpose of μ, the dominance partial order on partitions, the arm of c, the coarm of c, the leg of c, the coleg of c, and $\sum_{c \in \mu} l(c)$. Define $M = (1-q)(1-t)$, $B_\mu = \sum_{c \in \mu} q^{a'(c)} t^{l'(c)}$, $\Pi_\mu = \prod_{c \neq (0,0)} (1 - q^{a'(c)} t^{l'(c)})$, $T_\mu = q^{n(\mu')} t^{n(\mu)}$, and $w_\mu = \prod_{c \in \mu} [(q^{a(c)} - t^{l(c)+1})(t^{l(c)} - q^{a(c)+1})]$.

2. Symmetric Functions. We will work in the vector space Λ_F^n consisting of symmetric functions homogeneous of degree n in the variables $z_1, \ldots, z_n$ with coefficients in $F = \mathbb{Q}(q,t)$. We write m_μ, h_μ, e_μ, p_μ, and s_μ to denote (respectively) the monomial, homogeneous, elementary, power-sum, and Schur symmetric functions indexed by $\mu \vdash n$. These all form bases of Λ_F^n. The *Hall scalar product* is defined by requiring that the Schur basis be orthonormal. The power-sum basis is orthogonal relative to this scalar product, while the monomial basis is dual to the homogeneous basis.

3. Macdonald Polynomials and Nabla. The *modified Macdonald polynomials* [12, 15, 16, 25] form another basis for Λ_F^n. Using plethystic notation, these polynomials can be defined as the unique elements $\tilde{H}_\mu \in \Lambda_F^n$ such that $\tilde{H}_\mu[X(1-q)] \in \sum_{\lambda \geq \mu} F s_\lambda$; $\tilde{H}_\mu[X(1-t)] \in \sum_{\lambda \geq \mu'} F s_\lambda$; and $\langle \tilde{H}_\mu, s_n \rangle = 1$.

For an explicit combinatorial description of $\tilde{H}_\mu$, consult [10, 12]. Theorem 2.4 in [7] gives the following expansion of e_n in terms of modified Macdonald polynomials: $e_n = \sum_{\mu \vdash n}(MB_\mu \Pi_\mu / w_\mu)\tilde{H}_\mu$.

The nabla operator, introduced by F. Bergeron and Garsia [1, 2, 3], is the unique F-linear map on Λ_F^n defined on the basis $\{\tilde{H}_\mu\}$ by $\nabla(\tilde{H}_\mu) = T_\mu \tilde{H}_\mu$. It follows that $\nabla(e_n) = \sum_{\mu \vdash n}(MT_\mu B_\mu \Pi_\mu / w_\mu)\tilde{H}_\mu$.

4. Diagonal Harmonics. For $n \geq 1$, the symmetric group S_n acts diagonally on the polynomial ring $R_n = \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$, which is doubly graded by total degree in the x-variables and in the y-variables. The *diagonal harmonics module DH_n* consists of all $f \in R_n$ such that $\sum_{i=1}^n \partial x_i^h \partial y_i^k f = 0$ for all h, k with $h + k \geq 1$. The *diagonal harmonic alternants DHA_n* is the submodule of DH_n consisting of all f such that $\pi f = \mathrm{sgn}(\pi)f$ for $\pi \in S_n$. We write $\mathrm{Hilb}(DH_n)$ and $\mathrm{Hilb}(DHA_n)$ for the Hilbert series of these doubly-graded S_n-modules, which are elements of $\mathbb{N}[q, t]$. Write $\mathrm{Frob}(DH_n)$ for the Frobenius series of DH_n; note that $\mathrm{Frob}(DH_n) \in \Lambda_F^n$.

5. Master Theorem for the q, t-Catalan Numbers: For $n \geq 1$, the following five elements of $\mathbb{Q}(q, t)$ are all equal (and are, therefore, elements of $\mathbb{N}[q, t]$):

(a) $\sum_{D \in \mathcal{D}_n} q^{\mathrm{area}(D)} t^{\mathrm{bounce}(D)}$ (Haglund's combinatorial formula [8])
(b) $\sum_{D \in \mathcal{D}_n} q^{\mathrm{dinv}(D)} t^{\mathrm{area}(D)}$ (Haiman's combinatorial formula)
(c) $\langle \nabla(e_n), s_{1^n} \rangle$ (nabla formula)
(d) $\sum_{\mu \vdash n} T_\mu^2 MB_\mu \Pi_\mu / w_\mu$ (Garsia-Haiman rational-function formula [7])
(e) $\mathrm{Hilb}(DHA_n)$ (representation-theoretical formula)

See [5, 6, 7, 17, 18] for the proof.

6. Hilbert Series Conjecture [14]: For $n \geq 1$, the following six elements of $\mathbb{Q}(q, t)$ are all equal (and are, therefore, elements of $\mathbb{N}[q, t]$):

(a) $\sum_{P \in \mathcal{P}_n} q^{\mathrm{dinv}(P)} t^{\mathrm{area}(P)}$ (first combinatorial formula)
(b) $\sum_{P \in \mathcal{P}_n} q^{\mathrm{area}(P)} t^{\mathrm{pmaj}(P)}$ (second combinatorial formula [21])
(c) $\langle \nabla(e_n), h_{1^n} \rangle$ (nabla formula)
(d) $\sum_{\mu \vdash n} \langle \tilde{H}_\mu, h_{1^n} \rangle T_\mu MB_\mu \Pi_\mu / w_\mu$ (Macdonald polynomial formula)
(e) $\sum_{\mu \vdash n+1} M\Pi_\mu B_\mu^{n+1} / w_\mu$ (rational-function formula)
(f) $\mathrm{Hilb}(DH_n)$ (representation-theoretical formula)

It is known that (a)=(b) [21, 23] and that (c)=(d)=(e)=(f) [9, 17, 18].

7. Frobenius Series Conjecture [13]: For $n \geq 1$, the following five elements of Λ_F^n are all equal (and are, therefore, Schur-positive):

(a) $\sum_{Q \in \mathcal{Q}_n} q^{\mathrm{dinv}(Q)} t^{\mathrm{area}(Q)} z_1^{c_Q(1)} \cdots z_n^{c_Q(n)}$ (combinatorial formula)
(b) $\nabla(e_n)$ (nabla formula)
(c) $\sum_{\mu \vdash n} \tilde{H}_\mu T_\mu MB_\mu \Pi_\mu / w_\mu$ (Macdonald polynomial formula)
(d) $\sum_{\lambda \vdash n} s_\lambda \left(\sum_{\mu \vdash n+1} M\Pi_\mu B_\mu s_{\lambda'}[B_\mu] / w_\mu \right)$ (Schur expansion)
(e) $\mathrm{Frob}(DH_n)$ (representation-theoretical formula)

We know (b)=(c)=(d)=(e) [9, 17, 18], and (a) is symmetric in the z_i's [13].

2. Combinatorics of Square q,t-Lattice Paths

2.1. **Statistics for Square Paths.** A *square lattice path of order n* is a lattice path in an $n \times n$ square. Let $\mathcal{SQ}_n$ denote the set of square lattice paths of order n. We now define three statistics on paths in $\mathcal{SQ}_n$ generalizing the area, dinv, and bounce statistics defined in §1.1.

1. Square area *Statistic.* For $S \in \mathcal{SQ}_n$, let $\ell = \ell(S)$ be the minimum possible value such that S stays weakly above the line $y = x - \ell$. We call ℓ the *deviation* of the path S. Since S begins at the origin and ends at (n,n), we see that $0 \le \ell \le n$. Define the *area vector* $g(S) = (g_0(S), \ldots, g_{n-1}(S))$ by requiring that $g_i(S) + n - i$ be the number of complete boxes in the i'th row from the bottom between S and the line $x = n$. Note that the entries of this vector can be negative, but that this area vector reduces to the area vector in §1.1 when S is a Dyck path. We define $\mathrm{area}(S) = \sum_{i=0}^{n-1}(\ell + g_i(S))$. This can be interpreted as the number of complete boxes to the right of S and to the left of the line $y = x - \ell$.

2. Square dinv *Statistic.* Suppose $S \in \mathcal{SQ}_n$ has $(g_0(S), \ldots, g_{n-1}(S))$ as its area vector. Define

$$\mathrm{dinv}(S) = \sum_{i<j} \chi(g_i(S) - g_j(S) \in \{0,1\}) + \sum_i \chi(g_i(S) < -1).$$

If S is a Dyck path, then the condition $g_i(S) < -1$ never holds, and this formula for $\mathrm{dinv}(S)$ reduces to the formula given in §1.1.

3. Square bounce *Statistic.* Let $S \in \mathcal{SQ}_n$ have deviation ℓ. The *break point* of S, $(\ell_x(S), \ell_y(S))$, is the leftmost point along the path S lying on the line $y = x - \ell$.

We now proceed to define a bounce path $\mathrm{bpath}(S)$ in analogy with the bounce paths defined for Dyck paths. The bounce path for S consists of two pieces: a *positive part* located northeast of the break point, and a *negative part* located southwest of the break point. First consider the positive part. A ball starts at (n,n) and makes an initial vertical move V_{-1} of length $v_{-1} = \ell$ ending at $(n, n-\ell)$. The ball then makes alternating horizontal and vertical moves $H_0, V_0, H_1, V_1, \ldots, H_s, V_s$ until it reaches the break point. We let h_i and v_i denote the length of the moves H_i and V_i, respectively. We determine h_i and v_i for each $i \ge 0$ as follows. First, the ball moves west h_i units until it is blocked by the north step of S ending at the horizontal level occupied by the ball. Second, the ball moves south $v_i = h_i$ units to return to the line $y = x - \ell$. As before, the steps that block the ball's westward motion are called *blocking north steps*.

The negative part of the bounce path traces the motion of a second bouncing ball that starts at the origin and moves northeast towards the break point. This ball makes an initial horizontal move H_{-1} of length $h_{-1} = \ell$ from $(0,0)$ to $(\ell, 0)$. It then makes alternating vertical and horizontal moves $V_{-2}, H_{-2}, V_{-3}, H_{-3}, \ldots, V_u, H_u$ until it reaches the break point. For each

$i < -1$, the ball moves north v_i units until it is blocked by the east step of S *ending* at the vertical line occupied by the ball. (Note that this is *not* just a reflected version of the bounce algorithm in the positive part.) The ball then moves east $h_i = v_i$ units to return to the line $y = x - \ell$. The east steps that block the ball's northward motion are called *blocking east steps.*

Finally, we define the *bounce statistic* for any path $S \in \mathcal{SQ}_n$. Let $V_u, \ldots, V_s$ be the nonzero vertical moves in bpath(S), where $u \le 0 \le s$. Set bounce$(S) = \sum_{i=u}^{s}(i - u)v_i$. Also set bmin$(S) = u$ and bmax$(S) = s$.

For a Dyck path D, the deviation ℓ is 0, the break point is $(0,0)$, the positive part of the bounce path coincides with the bounce path described in §1.1, and the negative part of the bounce path is empty. In this case, we set bmin$(D) = 0$ (ignoring the empty moves V_{-1} and H_{-1}), and the bounce statistic just defined reduces to the formula used in §1.1.

For example, Figure 2 illustrates a path $S \in \mathcal{SQ}_{15}$ and its bounce path. For this path, $\ell(S) = 3$, the break point is $(8, 5)$,

$$g(S) = (0, -1, -2, -1, -1, -3, -2, -2, -2, -3, -2, -1, -1, 0, 1),$$

area$(S) = 25$, dinv$(S) = 52$, bmin$(S) = -4$, bmax$(S) = 2$, $(v_{-4}, \ldots, v_2) = (h_{-4}, \ldots, h_2) = (1, 2, 2, 3, 3, 2, 2)$, and bounce$(S) = 49$.
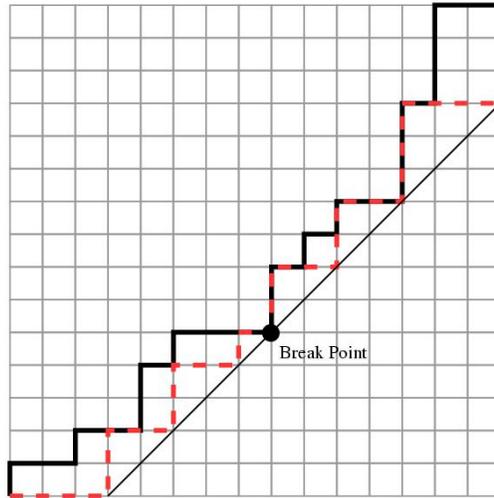

FIGURE 2. A path S in $\mathcal{SQ}_{15}$ (solid path) along with bpath(S) (dotted path).

2.2. Comparison of the Statistics.

Theorem 1. *There is a bijection $\phi : \mathcal{SQ}_n \to \mathcal{SQ}_n$ such that* area$(\phi(S)) =$ dinv(S) *and* bounce$(\phi(S)) =$ area(S). *The deviation of $\phi(S)$ is the number*

of -1's in $g(S)$; the break point of $\phi(S)$ is

$$(\ell_x(\phi(S)), \ell_y(\phi(S))) = (|\{j : g_j(S) < 0\}|, |\{j : g_j(S) < -1\}|);$$

bmin$(\phi(S)) = \min_j g_j(S)$; *and* bmax$(\phi(S)) = \max_j g_j(S)$. *Moreover, $\phi(S)$ ends with an east step iff S begins with a north step.*

Proof. Fix $S \in \mathcal{SQ}_n$ with deviation ℓ. We begin our construction of $\phi(S)$ by specifying its bounce path. Let u be the minimal value appearing in $g(S)$, and let s be the maximal value appearing in $g(S)$. It is easy to see that $u = -\ell$. For $u \le i \le s$, define v_i to be the number of $g_j(S)$'s equal to i, and define $h_i = v_i$. These data uniquely determine a bounce path $B(S)$ according to the rules above. We will construct $\phi(S)$ so that bpath$(\phi(S)) = B(S)$, $h_i(\phi(S)) = h_i$, and $v_i(\phi(S)) = v_i$.

It suffices to specify how the north and east steps of $\phi(S)$ are interleaved between successive bounces of $B(S)$. The ordering of the steps west of V_i and north of H_{i+1} is determined by the relative order of the entries i and $i + 1$ in the area vector for S. However, the details depend on whether we are considering the positive or negative part of the bounce path.

First, we identify the breakpoint of $\phi(S)$ as the point $(\sum_{j<0} h_j, \sum_{j<-1} v_j)$. The path $\phi(S)$ must pass through this point. Also, note that $\phi(S)$ runs east for h_u steps just before the breakpoint and north for v_s steps just after the breakpoint.

Second, for each $i \ge 0$, we describe the subpath of $\phi(S)$ located north of $H_i(\phi(S))$ and west of $V_{i-1}(\phi(S))$. Our description starts with the step just after (northeast of) the blocking north step that terminates the horizontal bounce H_i. Scan $g(S)$ *from right to left*, going north in $\phi(S)$ when we encounter a $g_j(S)$ equal to $i - 1$, and going east in $\phi(S)$ when we encounter a $g_j(S)$ equal to i. Note that this process generates $v_i = h_i$ east steps and v_{i-1} north steps, so the subpath ends where it should. Moreover, for $i > 0$, the last step generated by this scan must be a north step, since $g_0(S) \le 0$ and $g_{j+1}(S) \le g_j(S)+1$ imply that the leftmost i in $g(S)$ must be immediately preceded by an $i - 1$. This last step is the blocking north step that terminates the horizontal bounce H_{i-1}. On the other hand, for $i = 0$, the subpath ending at (n, n) may or may not end in a north step, and this fact does not influence the bounce path. Note that this last subpath ends in an east step if $g_0(S) = 0$. Conversely, if $g_0(S) < 0$, then the leftmost -1 in $g(S)$ must occur to the left of the leftmost 0, so that the last subpath ends in a north step. Since $g_0(S) = 0$ iff S begins with a north step, these comments show that $\phi(S)$ *ends in an east step iff S begins with a north step.* (If S is a Dyck path, there is no subpath corresponding to $i = 0$. In this case, $\phi(S)$ is also a Dyck path, and the italicized statement is obviously true. The restricted map $\phi|_{\mathcal{D}_n} : \mathcal{D}_n \to \mathcal{D}_n$ coincides with the bijection ϕ mentioned in §1.1.) For all paths S, it follows that the positive part of the bounce path for $\phi(S)$ coincides with the positive part of $B(S)$, as desired.

Third, for each $i < 0$, we describe the subpath of $\phi(S)$ located north of $H_i(\phi(S))$ and west of $V_{i-1}(\phi(S))$. Our description starts at the step

just after (northeast of) the blocking east step that terminates the vertical bounce V_i, or at $(0,0)$ for $i = -1$. Scan $g(S)$ from right to left, *ignoring* the first occurrence of a $g_j(S)$ equal to i. Continuing the scan, we go north in $\phi(S)$ when we encounter a $g_j(S)$ equal to $i - 1$, and we go east in $\phi(S)$ when we encounter a $g_j(S)$ equal to i. At the end of the scan, we append one more east step to $\phi(S)$, which is the blocking east step terminating the vertical bounce V_{i-1}. As before, this process guarantees that the negative part of bpath$(\phi(S))$ equals the negative part of $B(S)$.

It is easily checked that ϕ is invertible. To compute $T = \phi^{-1}(S')$, first draw bounce(S'). Next, for each i, use the subpaths of S' between the blocking steps to recover the substrings of $g(T)$ consisting of i's and $i - 1$'s. This is done by reversing the two scanning procedures just described for the positive and negative regions of the bounce path. Finally, these substrings suffice to determine $g(T)$ (and hence T) uniquely, because of the condition $g_{j+1}(T) \leq g_j(T) + 1$.

Finally, we check that ϕ has the desired effect on statistics. By construction, the bounce path of $\phi(S)$ is indeed $B(S)$, and the break point does lie at the coordinates given earlier. The statements in the theorem giving the deviation of $\phi(S)$ and the coordinates of the break point of $\phi(S)$ now follow from the formula $v_i = h_i = |\{j : g_j(S) = i\}|$. The same formula shows that

$$\text{bounce}(\phi(S)) = \sum_{i=u}^{s}(i - u)v_i = \sum_{i=u}^{s}(i + \ell)v_i = \sum_{j=0}^{n-1}(\ell + g_j(S)) = \text{area}(S).$$

Next, recall that area$(\phi(S))$ can be interpreted as the number of complete lattice squares east of $\phi(S)$ and west of the line $y = x - v_{-1}$, where v_{-1} is the deviation of $\phi(S)$. These cells can be partitioned into the following disjoint sets:

- the set B_i of cells east of V_i and west of $y = x - v_{-1}$, for all i;
- the set P_i of cells east of $\phi(S)$, north of H_i, and west of V_{i-1}, for all $i \geq 0$;
- the set N_i of cells east of $\phi(S)$, north of H_i, and west of V_{i-1}, for all $i < 0$.

Using the definition of ϕ, the following identities are easily verified:

$$|B_i| = \binom{v_i}{2} = \sum_{j<k} \chi(g_j(S) = g_k(S) = i);$$

$$|P_i| = \sum_{j<k} \chi(g_j(S) = i, \ g_k(S) = i - 1);$$

$$|N_i| = \sum_{j<k} \chi(g_j(S) = i, \ g_k(S) = i - 1) + \sum_{k} \chi(g_k(S) = i - 1).$$

The last summand in the formula for $|N_i|$ accounts for the area cells above bounce$(\phi(S))$ in the column below the blocking east step that terminates

V_{i-1}. Adding these formulas over all i, we see that $\mathrm{dinv}(S) = \mathrm{area}(\phi(S))$.

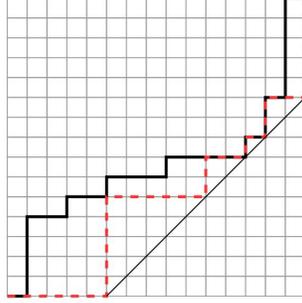<div style="text-align: right">□</div>

FIGURE 3. Image of the path S under ϕ.

For example, let S be the path from Figure 2. Figure 3 shows $\phi(S)$ along with its bounce path. Note that $g(S)$ has two -3's, five -2's, five -1's, two 0's, and one 1, so that

$$(v_{-3}(\phi(S)), \dots, v_1(\phi(S))) = (2, 5, 5, 2, 1).$$

Furthermore, $\mathrm{area}(S) = 25 = \mathrm{bounce}(\phi(S))$ and $\mathrm{dinv}(S) = 52 = \mathrm{area}(\phi(S))$.

2.3. **Symmetry Properties.** For all $n \geq 1$, define

$$S_n(q, t) = \sum_{S \in \mathcal{SQ}_n} q^{\mathrm{area}(S)} t^{\mathrm{bounce}(S)}.$$

Using the bijection ϕ, it follows that

$$S_n(q, t) = \sum_{S \in \mathcal{SQ}_n} q^{\mathrm{dinv}(S)} t^{\mathrm{area}(S)}.$$

Letting $q = 1$ or $t = 1$ here, we obtain the following *univariate symmetries*.

Corollary 2.

$$\sum_{S \in \mathcal{SQ}_n} q^{\mathrm{dinv}(S)} = \sum_{S \in \mathcal{SQ}_n} q^{\mathrm{bounce}(S)} = \sum_{S \in \mathcal{SQ}_n} q^{\mathrm{area}(S)}.$$

Conjecture 3. The *joint symmetry* $S_n(q, t) = S_n(t, q)$ holds for all n.

This conjecture has been confirmed by computer for $1 \leq n \leq 11$.

Computing $S_n(q, t)$ for small values of n, one sees that the polynomial $S_n(q, t)$ is always divisible by 2. Our next goal is to explain this property. Define $\mathcal{SQ}_n^N$, $\mathcal{SQ}_n^E$, $^N\mathcal{SQ}_n$, and $^E\mathcal{SQ}_n$ to be the paths in $\mathcal{SQ}_n$ that end with

a north step, end with an east step, begin with a north step, and begin with an east step, respectively. Set

$$\begin{aligned}
S_n^N(q,t) &= \sum_{S \in \mathcal{SQ}_n^N} q^{\text{area}(S)} t^{\text{bounce}(S)}; \\
S_n^E(q,t) &= \sum_{S \in \mathcal{SQ}_n^E} q^{\text{area}(S)} t^{\text{bounce}(S)}; \\
{}^N S_n(q,t) &= \sum_{S \in {}^N\mathcal{SQ}_n} q^{\text{dinv}(S)} t^{\text{area}(S)}; \\
{}^E S_n(q,t) &= \sum_{S \in {}^E\mathcal{SQ}_n} q^{\text{dinv}(S)} t^{\text{area}(S)}.
\end{aligned}$$

We will show that $S_n^N(q,t) = S_n^E(q,t) = S_n(q,t)/2$. Since ϕ sends paths with initial north steps to paths with terminal east steps and vice versa, it also follows that ${}^N S_n(q,t) = {}^E S_n(q,t) = S_n(q,t)/2$. We call these identities *pair-symmetries*.

To prove the pair-symmetries, it suffices to construct a bijection $\psi :$ $\mathcal{SQ}_n^E \to \mathcal{SQ}_n^N$ preserving area and bounce. We begin by introducing a cyclic shift map $\text{cyc} : \mathcal{SQ}_n \to \mathcal{SQ}_n$. Let $S \in \mathcal{SQ}_n$ be encoded by the word $w_1 w_2 \cdots w_{2n} \in \{0,1\}^{2n}$. Define $\text{cyc}(S)$ to be the path encoded by the word $w_{2n} w_1 w_2 \cdots w_{2n-1}$.

Lemma 4. *For $S \in \mathcal{SQ}_n$, $\text{area}(S) = \text{area}(\text{cyc}^i(S))$ for all integers i.*

Proof. It suffices to prove the equality for $i = 1$. Let $S' = \text{cyc}(S)$, and let the deviation of S be ℓ. First consider the case where the last step of S is a north step. Then $\ell \geq 1$ and the deviation of $\text{cyc}(S)$ is $\ell' = \ell - 1$. Furthermore, $g_0(S') = 0$, $g_{n-1}(S) = -1$, and $g_j(S') = g_{j-1}(S) + 1$ for $1 \leq j < n$. So, by definition,

$$\begin{aligned}
\text{area}(S') &= (\ell' + g_0(S')) + \sum_{j=1}^{n-1} (\ell' + g_j(S')) \\
&= (\ell - 1 + 0) + \sum_{j=0}^{n-2} (\ell - 1 + g_j(S) + 1) \\
&= \sum_{j=0}^{n-1} (\ell + g_j(S)) = \text{area}(S).
\end{aligned}$$

Now suppose S ends in an east step. In this case, $\ell' = \ell + 1$, and $g_j(S') = g_j(S) - 1$ for each $0 \leq j < n$. It follows immediately that $\text{area}(S') = \text{area}(S)$ for this case as well. $\qquad\square$

Theorem 5. *There is a bijection $\psi : \mathcal{SQ}_n^E \to \mathcal{SQ}_n^N$ preserving* area *and* bounce. *Consequently,*

$$S_n^N(q,t) = S_n^E(q,t) = S_n(q,t)/2 = {}^N S_n(q,t) = {}^E S_n(q,t).$$

Proof. Let $S \in \mathcal{SQ}_n^E$. Let A denote the east step at the end of the path S, and let B denote the blocking north step terminating the initial horizontal move H_0 (of length h_0) in the positive part of the bounce path for S. Define $S' = \psi(S)$ to be $\text{cyc}^{v_{-1}(S)+h_0(S)}(S)$, and let A' and B' denote the steps corresponding to A and B under this cyclic shift. Since the north step B' is the last step of S', we have $S' \in \mathcal{SQ}_n^N$. By the lemma, $\text{area}(S') = \text{area}(S)$. Let ℓ and ℓ' be the deviations of S and S', respectively. Note that $\ell = v_{-1}(S)$. Recall that applying cyc to a path in $\mathcal{SQ}_n^E$ increases the deviation by 1, while applying cyc to a path in $\mathcal{SQ}_n^N$ decreases the deviation by 1. Therefore, $v_{-1}(S') = \ell' = \ell + h_0(S) - v_{-1}(S) = h_0(S) = v_0(S)$. Now, the bounce $V_0(S)$ begins at B, while the bounce $V_{-1}(S')$ begins at B'. Since $v_{-1}(S') = v_0(S)$, it follows easily that $v_i(S') = v_{i+1}(S)$ for $-1 \le i < s$. See Figure 4 for an example.

FIGURE 4. Example of the bijection ψ.

In the negative part of the bounce path for S', we have $h_{-1}(S') = \ell' = h_0(S)$. It follows that A' will be the blocking east step that terminates $V_{-2}(S')$, and hence $v_{-2}(S') = v_{-1}(S)$. Tracing the bounce paths upward, we easily see that $v_i(S') = v_{i+1}(S)$ for $u - 1 \le i \le -2$. Therefore,

$$\text{bounce}(S') = \sum_{i=u-1}^{s-1} (i - (u-1))v_i(S') = \sum_{i=u-1}^{s-1} (i+1-u)v_{i+1}(S)$$

$$= \sum_{i=u}^{s} (i-u)v_i(S) = \text{bounce}(S).$$

Finally, ψ is a bijection: to invert it, we just cyclically shift S' backwards so that the first blocking east step A' becomes the final east step of the path. $\square$

We close this section with an alternate formula for ${}^E S_n(q,t) = S_n(q,t)/2$. Define $\text{dinv}_0(S) = \sum_{i<j} \chi(g_i(S) - g_j(S) \in \{0,1\}) + \sum_i \chi(g_i(S) < 0)$.

Theorem 6. *For all $n \geq 1$,*

$$^{E}S_n(q,t) = \sum_{S \in {^{E}\mathcal{SQ}_n}} q^{\mathrm{dinv}(S)} t^{\mathrm{area}(S)} = \sum_{U \in \mathcal{SQ}_n^E} q^{\mathrm{dinv}_0(U)} t^{\mathrm{area}(U)}.$$

Proof. The map cyc^{-1} is a bijection $^{E}\mathcal{SQ}_n \to \mathcal{SQ}_n^E$ that preserves area. Letting $U = \mathrm{cyc}^{-1}(S)$, we see that $\mathrm{area}(U) = \mathrm{area}(S)$ and $g_i(U) = g_i(S) + 1$ for all i. This implies that $g_i(U) - g_j(U) = g_i(S) - g_j(S)$ for all $i < j$, while $g_i(U) < 0$ iff $g_i(S) < -1$. Hence, $\mathrm{dinv}_0(U) = \mathrm{dinv}(S)$, and the theorem follows. $\square$

2.4. **Recursion for Square Paths.** In §1.1, we saw that the generating functions $C_{n,k}(q,t)$ for Dyck paths of order n with $h_0 = k$ satisfied the recursion (3). Now we prove a similar recursion for square q, t-lattice paths. The idea is to remove the "earliest" bounce in a square q, t-lattice path, namely the negative bounce arriving at the break point.

Formally, for $n > 0$ and $1 \leq k \leq n$, we set

$$R_{n,k}(q,t) = \sum_{S \in \mathcal{SQ}_n} q^{\mathrm{area}(S)} t^{\mathrm{bounce}(S)} \chi(h_{\mathrm{bmin}(S)} = k, \ \ell(S) > 0).$$

The condition $\ell(S) > 0$ means that S is not a Dyck path, while $h_{\mathrm{bmin}(S)} = k$ means that the last horizontal move in the negative part of the bounce path (arriving at the break point) has length k. To take care of the Dyck paths in $\mathcal{SQ}_n$, we define $R_{n,0}(q,t) = C_n(q,t) = t^{-n} C_{n+1,1}(q,t)$ for $n \geq 0$. For $k = n \geq 0$, we have $R_{n,n}(q,t) = q^{\binom{n}{2}}$ since the only path that contributes is the one that goes east n steps and then north n steps. Clearly, $S_n(q,t) = \sum_{k=0}^{n} R_{n,k}(q,t)$.

Theorem 7. *For $0 < k < n$,*

$$(8) \quad R_{n,k}(q,t) = q^{\binom{k}{2}} t^{n-k} \sum_{r=1}^{n-k} \begin{bmatrix} r+k \\ r, k \end{bmatrix}_q C_{n-k,r}(q,t)$$

$$+ q^{\binom{k}{2}} t^{n-k} \sum_{r=1}^{n-k} q^k \begin{bmatrix} r+k-1 \\ r-1, k \end{bmatrix}_q R_{n-k,r}(q,t).$$

Proof. Any path S contributing to $R_{n,k}$ either has its break point on the bottom row (i.e., $\ell_y = 0$), or has the break point above the bottom row (i.e., $\ell_y > 0$). These two cases will correspond to the two summations on the right side of (8).

First consider the case where the break point is on the bottom row. This implies that S is encoded by $1^k w_{k+1} \cdots w_{2n}$. Letting $r = h_0(S)$, so that $1 \leq r \leq n - k$, it follows from our description of bouncing for square lattice paths that the northeast end of $\mathrm{bpath}(S)$ is encoded by $01^r 0^k$, regardless of what S does in its last $r + k$ steps. Hence, the path D encoded by $w_{k+1} w_{k+2} \cdots w_{2n-r-k} 1^r$ is a Dyck path of order $n - k$ with

$h_0(D) = r = h_0(S)$. This accounts for the factor $C_{n-k,r}(q,t)$. The factor $\begin{bmatrix} r+k \\ r,k \end{bmatrix}_q$ corresponds to choosing the subpath S' consisting of the final $r + k$ steps of S. The power of q in this factor records $\mathrm{ar}(S')$, which is the number of area squares between S' and $\mathrm{bpath}(S)$. We have now obtained the summand in the first summation of (8). The total power of q is correct, thanks to the identity $\mathrm{area}(S) = \binom{k}{2} + \mathrm{area}(D) + \mathrm{ar}(S')$. Moreover, we have $\mathrm{bmin}(S) = -1$, $\mathrm{bmin}(D) = 0$, $v_{-1}(S) = k$, and $v_i(D) = v_i(S)$ for $0 \le i \le \mathrm{bmax}(S) = \mathrm{bmax}(D)$. Therefore,

$$\mathrm{bounce}(S) = \sum_{i=0}^{\mathrm{bmax}(S)} (i+1)v_i(S) = \sum_{i=0}^{\mathrm{bmax}(D)} iv_i(D) + \sum_{i \ge 0} v_i(S)$$
$$= \mathrm{bounce}(D) + n - k,$$

so the power of t is correct as well. Note that every triple (r, S', D) counted by the first summation arises from a unique path S in the manner described.

Now consider the case where the break point does not lie on the bottom row. We then have $u = \mathrm{bmin}(S) \le -2$, $v_u(S) = h_u(S) = k$, and $v_{u+1}(S) = h_{u+1}(S) = r$ for some $r > 0$. Let S' be the subpath of S starting just after the blocking east step that terminates V_{u+1} and ending just before the blocking east step that terminates V_u. Note that S' is an arbitrary lattice path consisting of $r - 1$ east steps and k north steps. This accounts for the term $\begin{bmatrix} r+k-1 \\ r-1,k \end{bmatrix}_q$ in the second summation.

Next, let S_0 be obtained from S by deleting S' and the $k + 1$ east steps following S', and replacing them by r east steps. (Intuitively, we are excising the k rows of the figure immediately below the break point, and then sliding the break point k units down and k units left along the line $y = x - \ell(S)$.) One checks easily that S_0 is one of the paths enumerated by $R_{n-k,r}(q,t)$, that $\mathrm{bmin}(S_0) = u + 1$, and that $h_i(S_0) = v_i(S_0) = v_i(S) = h_i(S)$ for $\mathrm{bmin}(S_0) \le i \le \mathrm{bmax}(S_0) = \mathrm{bmax}(S)$. As above, it easily follows that $\mathrm{bounce}(S) = \mathrm{bounce}(S_0) + n - k$. Moreover, since $\mathrm{ar}(S')$ is the number of area squares below S' and above $\mathrm{bpath}(S)$, we see that $\mathrm{area}(S) = \mathrm{area}(S_0) + \mathrm{ar}(S') + \binom{k}{2} + k$. Here, the k added at the end accounts for the extra area cells below the blocking east step immediately following S'. Since the passage from S and k to the triple (r, S', S_0) is reversible, the proof of the recursion is complete. $\qquad\square$

Note that recursions (3) and (8), and the initial conditions, uniquely determine the quantities $R_{n,k}(q,t)$ and $S_n(q,t)$ and provide an efficient method for computing them.

2.5. Fermionic Formula.
We now obtain a fermionic formula for $S_n(q,t)$ in analogy with (4).

Theorem 8. *For $n \geq 1$,*

$$(9) \quad S_n(q,t) = 2q^{\binom{n}{2}} + \sum_{\substack{w_0 + \cdots + w_s = n \\ w_i > 0; s \geq 1}} \left(q^{p_1} t^{p_2} \prod_{j=0}^{s-1} \begin{bmatrix} w_j + w_{j+1} - 1 \\ w_j - 1, w_{j+1} \end{bmatrix}_q + \right.$$

$$\left. \sum_{a=0}^{s-1} q^{p_3} t^{p_2} \begin{bmatrix} w_a + w_{a+1} \\ w_a, w_{a+1} \end{bmatrix}_q \prod_{j=0}^{a-1} \begin{bmatrix} w_j + w_{j+1} - 1 \\ w_j, w_{j+1} - 1 \end{bmatrix}_q \prod_{j=a+1}^{s-1} \begin{bmatrix} w_j + w_{j+1} - 1 \\ w_j - 1, w_{j+1} \end{bmatrix}_q \right)$$

where $p_1 = \sum_{j=0}^{s} \binom{w_j}{2}$, $p_2 = \sum_{j=0}^{s} j w_j$, and $p_3 = p_1 + \sum_{0 \leq j < a} w_j$.

Proof. One can deduce this formula by simply examining the picture of a path S counted by $R_{n,k}$ and its bounce path bpath(S). The paths S_0 and S_1 encoded by $0^n 1^n$ and $1^n 0^n$ account for the initial term $2q^{\binom{n}{2}}$ in (9). Of the two primary summands in (9), the first gives the contributions of all Dyck paths other than S_0, as can be checked by comparing with (4). Here, $w_i = v_i(D)$ for $0 \leq i \leq s = \mathrm{bmax}(D)$, and the q-binomial coefficients account for the subpaths of D above bpath(D), keeping in mind the blocking north steps.

The second summand accounts for all non-Dyck paths S in $\mathcal{SQ}_n$ other than S_1. To translate a particular term in this summand into a path S, take $\mathrm{bmin}(S) = -(a+1)$, $\mathrm{bmax}(S) = \mathrm{bmin}(S) + s$, and $v_i(S) = w_{i-\mathrm{bmin}(S)}$ for $\mathrm{bmin}(S) \leq i \leq \mathrm{bmax}(S)$. In particular, $v_0(S) = w_{a+1}$ and $h_{-1}(S) = w_a$. One checks that p_3 counts the area squares between bpath(S) and $y = x - \ell(S)$, plus the extra area squares above bpath(S) in the columns below the blocking east steps. It is also clear that $p_2 = \mathrm{bounce}(S)$. The first q-binomial coefficient on the second line of (9) accounts for the subpath of S above the zeroth horizontal bounce move, which is an arbitrary lattice path with w_{a+1} east steps and w_a north steps. The product of q-binomial coefficients for $0 \leq j < a$ accounts for the subpaths of S above the bounces in the negative part of the bounce path, keeping in mind the blocking east steps. The product of q-binomial coefficients for $a < j < s$ accounts for the subpaths of S above the bounces in the positive part of the bounce path, keeping in mind the blocking north steps. $\square$

For $0 < k < n$, there is a similar fermionic formula for $R_{n,k}(q,t)$. We simply use the second line of (9), summing over all $(w_0, \ldots, w_s)$ and all a such that $w_0 + \cdots + w_s = n$, $w_i > 0$, $s \geq 1$, $0 \leq a \leq s - 1$, *and* fixing $w_0 = k$.

2.6. **Specialization at $t = 1/q$.** Our next goal is to derive explicit formulas for $R_{n,k}(q, 1/q)$ and $S_n(q, 1/q)$ similar to the formulas for $C_{n,k}(q, 1/q)$ and $C_n(q, 1/q)$ from §1.1. We shall need the following standard identities:

$$(10) \qquad \begin{bmatrix} c + d \\ c, d \end{bmatrix}_q = q^d \begin{bmatrix} c + d - 1 \\ c - 1, d \end{bmatrix}_q + \begin{bmatrix} c + d - 1 \\ c, d - 1 \end{bmatrix}_q;$$

(11)
$$\begin{bmatrix} c+d \\ c,d \end{bmatrix}_q = \begin{bmatrix} c+d-1 \\ c-1,d \end{bmatrix}_q + q^c \begin{bmatrix} c+d-1 \\ c,d-1 \end{bmatrix}_q;$$

(12)
$$\begin{bmatrix} c+d+e+1 \\ c,d+1+e \end{bmatrix}_q = \sum_{u=0}^{c} \begin{bmatrix} u+d \\ u,d \end{bmatrix}_q \begin{bmatrix} c+e-u \\ c-u,e \end{bmatrix}_q q^{u(e+1)};$$

(13)
$$\begin{bmatrix} c+d \\ c,d \end{bmatrix}_q = \sum_{v=0}^{d} q^{cv} \begin{bmatrix} c+d-v-1 \\ c-1,d-v \end{bmatrix}_q.$$

As in [24], the reader may give computation-free bijective proofs of these identities by drawing suitable pictures and using (1). For instance, (13) classifies paths in a $d \times c$ rectangle based on the number v of east steps following the final north step.

Theorem 9. *For* $1 \le k \le n$,

(14)
$$q^{\binom{n-k+1}{2}-\binom{k}{2}} R_{n,k}(q,1/q) = \begin{bmatrix} 2n-k-1 \\ n-1,n-k \end{bmatrix}_q + q^k \begin{bmatrix} 2n-k-1 \\ n,n-k-1 \end{bmatrix}_q.$$

For $k=0$, $q^{\binom{n}{2}} R_{n,0}(q,1/q) = q^{\binom{n}{2}} C_n(q,1/q)$ *is given by the formulas*

$$\begin{bmatrix} 2n \\ n,n \end{bmatrix}_q - q \begin{bmatrix} 2n \\ n-1,n+1 \end{bmatrix}_q = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n,n \end{bmatrix}_q.$$

Proof. Recall from §1.1 that

(15)
$$q^{\binom{u+1}{2}-uv} C_{u,v}(q,1/q) = \begin{bmatrix} 2u-v-1 \\ u-v,u-1 \end{bmatrix}_q - q^v \begin{bmatrix} 2u-v-1 \\ u-v-1,u \end{bmatrix}_q.$$

This identity can be proved from (3) by manipulations similar to those given in this proof; see [19, 24] for details. The formulas for $C_n(q,1/q) = R_{n,0}(q,1/q)$ now follow easily from the fact that $C_n(q,t) = t^{-n} C_{n+1,1}(q,t)$.

Now we prove (14) by induction on n, the cases $n \le 1$ and $n = k$ being easy. Using the induction hypothesis and (15) on the right side of (8), we see that $q^{\binom{n-k+1}{2}-\binom{k}{2}} R_{n,k}(q,1/q)$ equals

(16)
$$q^{p_1} \sum_{r=1}^{n-k} \left\{ \begin{bmatrix} r+k \\ r,k \end{bmatrix}_q q^{p_2} (\alpha - q^r \beta) + q^k \begin{bmatrix} r+k-1 \\ r-1,k \end{bmatrix}_q q^{p_3} (\alpha + q^r \beta) \right\},$$

where $\alpha = \begin{bmatrix} 2(n-k)-r-1 \\ n-k-r,n-k-1 \end{bmatrix}_q$, $\beta = \begin{bmatrix} 2(n-k)-r-1 \\ n-k-r-1,n-k \end{bmatrix}_q$, $p_1 = \binom{n-k+1}{2} - (n-k)$, $p_2 = (n-k)r - \binom{n-k+1}{2}$, and $p_3 = \binom{r}{2} - \binom{n-k-r+1}{2}$.

To continue simplifying (16), we proceed in seven steps. Step 1: we simplify the powers of q. One easily calculates that $p_1 + p_2 = (n-k)(r-1)$ and $p_1 + p_3 + k = n(r-1) - k(r-2)$. Step 2: we break (16) into several smaller sums. To do this, use (10) to rewrite $\begin{bmatrix} r+k \\ r,k \end{bmatrix}_q$ as $q^k \begin{bmatrix} r+k-1 \\ r-1,k \end{bmatrix}_q + \begin{bmatrix} r+k-1 \\ r,k-1 \end{bmatrix}_q$ and then expand the contents of the curly braces in (16) using the distributive law. What results is a sum of six terms. Two of these terms are equal, and

two others cancel. After cancelling and grouping, we can rewrite (16) as a sum $A + B + C$, where:

$$(17) \quad A = 2q^k \sum_{r=1}^{n-k} q^{(n-k)(r-1)} \begin{bmatrix} r+k-1 \\ r-1, k \end{bmatrix}_q \begin{bmatrix} 2n-2k-r-1 \\ n-k-r, n-k-1 \end{bmatrix}_q ;$$

$$(18) \quad B = q^{k-n} \sum_{r=1}^{n-k} q^{(n-k)r} \begin{bmatrix} r+k-1 \\ r, k-1 \end{bmatrix}_q \begin{bmatrix} 2n-2k-r-1 \\ n-k-r, n-k-1 \end{bmatrix}_q ;$$

$$(19) \quad C = -q^{k-n} \sum_{r=1}^{n-k} q^{(n-k+1)r} \begin{bmatrix} r+k-1 \\ r, k-1 \end{bmatrix}_q \begin{bmatrix} 2n-2k-r-1 \\ n-k-r-1, n-k \end{bmatrix}_q .$$

Step 3: we evaluate A. Using (12) with $c = n-k-1$, $d = k$, $e = n-k-1$, and $u = r-1$, we find that

$$A = 2q^k \begin{bmatrix} 2n-k-1 \\ n-k-1, n \end{bmatrix}_q .$$

Step 4: we evaluate B. We use (12) with $c = n-k$, $d = k-1$, $e = n-k-1$, and $u = r$. Since the $u = 0$ summand is missing in B, we must add and subtract it. We find that

$$B = q^{k-n} \left(\begin{bmatrix} 2n-k-1 \\ n-k, n-1 \end{bmatrix}_q - \begin{bmatrix} 2n-2k-1 \\ n-k, n-k-1 \end{bmatrix}_q \right) .$$

Step 5: we evaluate C. We use (12) with $c = n-k-1$, $d = k-1$, $e = n-k$, and $u = r$. Since the $u = 0$ summand is missing in C, we must add and subtract it. Also note that the $r = n-k$ term is 0. We find that

$$C = -q^{k-n} \left(\begin{bmatrix} 2n-k-1 \\ n-k-1, n \end{bmatrix}_q - \begin{bmatrix} 2n-2k-1 \\ n-k-1, n-k \end{bmatrix}_q \right) .$$

Step 6: we compute $B + C$. The second terms in the preceding formulas for B and C cancel; using (10) and (11), we find that

$$B + C = q^{k-n} \left(\begin{bmatrix} 2n-k-1 \\ n-k, n-1 \end{bmatrix}_q - \begin{bmatrix} 2n-k-1 \\ n-k-1, n \end{bmatrix}_q \right)$$

$$= \begin{bmatrix} 2n-k-1 \\ n-k, n-1 \end{bmatrix}_q - q^k \begin{bmatrix} 2n-k-1 \\ n-k-1, n \end{bmatrix}_q .$$

Step 7: We combine steps 3 and 6 to obtain

$$q^{\binom{n-k+1}{2}-\binom{k}{2}} R_{n,k}(q, 1/q) = A+B+C = \begin{bmatrix} 2n-k-1 \\ n-k, n-1 \end{bmatrix}_q + q^k \begin{bmatrix} 2n-k-1 \\ n-k-1, n \end{bmatrix}_q ,$$

completing the induction. $\qquad \square$

Theorem 10. *For all $n \geq 1$,*

$$q^{\binom{n}{2}} S_n(q, 1/q) = 2 \begin{bmatrix} 2n-1 \\ n, n-1 \end{bmatrix}_q = \frac{2}{1+q^n} \begin{bmatrix} 2n \\ n, n \end{bmatrix}_q .$$

Proof. Recall that $S_n(q, 1/q) = \sum_{k=0}^{n} R_{n,k}(q, 1/q)$. Using the formulas in the previous theorem for $R_{n,k}(q, 1/q)$, and writing $p_1 = \binom{n}{2} + \binom{k}{2} - \binom{n-k+1}{2}$, $p_2 = \binom{n-k+1}{2} - \binom{k}{2}$, we get

$$(20) \quad q^{\binom{n}{2}} S_n(q, 1/q) = q^{\binom{n}{2}} R_{n,0}(q, 1/q) + \sum_{k=1}^{n} q^{p_1} [q^{p_2} R_{n,k}(q, 1/q)]$$

$$(21) \qquad\qquad = q^{-n} \left\{ \begin{bmatrix} 2n \\ n, n \end{bmatrix}_q - \begin{bmatrix} 2n \\ n-1, n+1 \end{bmatrix}_q + D + E \right\}, \text{ where}$$

$$(22) \qquad D = \sum_{k=1}^{n} q^{nk} \begin{bmatrix} 2n-k-1 \\ n-1, n-k \end{bmatrix}_q,$$

$$(23) \qquad E = \sum_{k=1}^{n} q^{(n+1)k} \begin{bmatrix} 2n-k-1 \\ n, n-k-1 \end{bmatrix}_q.$$

We evaluate D by using (13) with $c = d = n$ and $v = k$ (adding and subtracting the missing $v = 0$ summand), which gives

$$D = \begin{bmatrix} 2n \\ n, n \end{bmatrix}_q - \begin{bmatrix} 2n \\ n-1, n \end{bmatrix}_q.$$

We evaluate E by using (13) with $c = n+1$, $d = n-1$ and $v = k$ (adding and subtracting the missing $v = 0$ summand), which gives

$$E = \begin{bmatrix} 2n \\ n+1, n-1 \end{bmatrix}_q - \begin{bmatrix} 2n \\ n, n-1 \end{bmatrix}_q.$$

Inserting these expressions into (21) and simplifying, we get

$$q^{\binom{n}{2}} S_n(q, 1/q) = 2q^{-n} \left(\begin{bmatrix} 2n \\ n, n \end{bmatrix}_q - \begin{bmatrix} 2n-1 \\ n, n-1 \end{bmatrix}_q \right).$$

Finally, since $\begin{bmatrix} 2n \\ n,n \end{bmatrix}_q = q^n \begin{bmatrix} 2n-1 \\ n-1,n \end{bmatrix}_q + \begin{bmatrix} 2n-1 \\ n,n-1 \end{bmatrix}_q$ by (10), we see that

$$q^{\binom{n}{2}} S_n(q, 1/q) = 2 \begin{bmatrix} 2n-1 \\ n-1, n \end{bmatrix}_q.$$

The identity $2 \begin{bmatrix} 2n-1 \\ n,n-1 \end{bmatrix}_q = 2 \begin{bmatrix} 2n \\ n,n \end{bmatrix}_q / (1 + q^n)$ follows from a simple algebraic manipulation. $\qquad\qquad\square$

We remark that the methods in [24] can be used to mechanically translate the preceding algebraic manipulations into bijective proofs of the same results. However, because of all the subtractions involved, the bijections will be extremely complicated.

3. ALGEBRAIC CONJECTURES FOR SQUARE PATHS

We now give some conjectures connecting square q, t-lattice paths to Macdonald polynomials and the nabla operator.

3.1. **Unlabelled Paths.** *Master Conjecture for Square q,t-Lattice Paths:*
For all $n \geq 1$, the following elements of $\mathbb{Q}(q,t)$ are all equal (and are, there-
fore, elements of $\mathbb{N}[q,t]$):

(a) $\sum_{S \in \mathcal{SQ}_n^N} q^{\text{area}(S)} t^{\text{bounce}(S)}$
(b) $\sum_{S \in \mathcal{SQ}_n^E} q^{\text{area}(S)} t^{\text{bounce}(S)}$
(c) $\sum_{S \in {}^N\mathcal{SQ}_n} q^{\text{dinv}(S)} t^{\text{area}(S)}$
(d) $\sum_{S \in {}^E\mathcal{SQ}_n} q^{\text{dinv}(S)} t^{\text{area}(S)}$
(e) $\sum_{S \in \mathcal{SQ}_n^E} q^{\text{dinv}_0(S)} t^{\text{area}(S)}$
(f) $(-1)^{n-1} \langle \nabla(p_n), s_{1^n} \rangle$
(g) $\sum_{\mu \vdash n} (1 - t^n)(1 - q^n) \Pi_\mu T_\mu^2 / w_\mu = \sum_{\mu \vdash n} MB_{(n^n)} \Pi_\mu T_\mu^2 / w_\mu$

We have already seen that (a) through (e) are equal, using the bijections
ψ, ϕ, and cyc^{-1}. To see that (f) equals (g), we use the expansion $p_n = \sum_{\mu \vdash n} ((-1)^{n-1}(1 - t^n)(1 - q^n) \Pi_\mu / w_\mu) \tilde{H}_\mu$, which follows immediately from
Corollary 2.4 in [7] and the definition of plethysm. Applying ∇ gives

$$(-1)^{n-1} \nabla(p_n) = \sum_{\mu \vdash n} ((1 - t^n)(1 - q^n) \Pi_\mu T_\mu / w_\mu) \tilde{H}_\mu.$$

Taking the scalar product with s_{1^n} turns $\tilde{H}_\mu$ into another factor T_μ. Hence,
(f) equals (g). The main conjecture ((a)=(f)) has been tested for $1 \leq n \leq 8$.

3.2. **Labelled Paths.** Fix n and N with $n \leq N \leq \infty$. Let $\mathcal{SQF}_n$ denote
the set of all pairs (S, r), where: S is a path in $\mathcal{SQ}_n^E$ (so that S ends with an
east step); and $r = r_0 \ldots r_{n-1}$ is a *label vector* with $r_i \in \{1, 2, \ldots, N\}$ such
that $g_{i+1}(S) = g_i(S) + 1$ implies $r_i < r_{i+1}$. If we attach the labels r_i to the
vertical steps of S as we did for Dyck paths, then the last condition means
that labels in each column must strictly increase from bottom to top. Let
$\mathcal{SQH}_n$ denote the subset of $\mathcal{SQF}_n$ such that $r_0 \ldots r_{n-1}$ is a permutation of
$\{1, 2, \ldots, n\}$. Given $(S, r) \in \mathcal{SQF}_n$, define $\text{area}(S, r) = \text{area}(S)$ and

$$\text{dinv}_0(S, r) = \sum_{i < j} \chi \left((g_i(S) - g_j(S) = 0 \text{ and } r_i < r_j) \text{ or} \right.$$

$$\left. (g_i(S) - g_j(S) = 1 \text{ and } r_i > r_j) \right) + \sum_{i=0}^{n-1} \chi(g_i(S) < 0).$$

(It is equivalent to use all labelled paths *beginning* with an east step, replac-
ing $\chi(g_i(S) < 0)$ by $\chi(g_i(S) < -1)$ in the definition of dinv_0.)

Hilbert series conjecture for square q,t-lattice paths: For all $n \geq 1$,

$$(-1)^{n-1} \langle \nabla(p_n), h_{1^n} \rangle = \sum_{(S,r) \in \mathcal{SQH}_n} q^{\text{area}(S,r)} t^{\text{dinv}_0(S,r)}.$$

Frobenius series conjecture for square q,t-lattice paths: For all $n \geq 1$,

$$(-1)^{n-1}\nabla(p_n[z_1,\ldots,z_N]) = \sum_{(S,r)\in\mathcal{SQF}_n} q^{\mathrm{area}(S,r)} t^{\mathrm{dinv}_0(S,r)} \prod_{i=0}^{n-1} z_{r_i}.$$

We remark that the same arguments used in [13] show that the Frobenius series conjecture implies the Hilbert series conjecture and the master conjecture for unlabelled square paths, along with shuffle-type formulas for any scalar product of the form $\langle\nabla(p_n), h_\mu e_\eta\rangle$. It is an open problem to find a naturally occurring doubly-graded S_n-module M_n that has $(-1)^{n-1}\nabla(p_n)$ as its Frobenius series. Since elements of $\mathcal{SQH}_n$ encode functions $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ in the obvious way ($f^{-1}(\{i\})$ is the set of labels in column i), we should have $\dim(M_n) = |\mathcal{SQH}_n| = n^n$.

3.3. **More Nabla Conjectures.** So far, we have seen combinatorial formulas that are conjectured to give the monomial expansions of $\nabla(e_n)$ and $(-1)^{n-1}\nabla(p_n)$. Since $e_n = m_{(1^n)}$ and $p_n = m_{(n)}$, these results suggest that $\nabla(m_\mu)$ may have a nice monomial expansion for any $\mu \vdash n$. In fact, an even stronger statement appears to be true.

Conjecture 11. For all $n \geq 1$ and $\mu, \nu \vdash n$,

$$(-1)^{n-\ell(\mu)}\langle\nabla(m_\mu), s_\nu\rangle \in \mathbb{N}[q,t].$$

The conjecture has been tested for $1 \leq n \leq 8$. If the conjecture is true, it readily follows that $\nabla(m_\mu)|_{m_\nu} \in \mathbb{N}[q,t]$ for all μ and ν. In [3], Bergeron, Garsia, Haiman, and Tesler made the analogous conjecture

$$\iota(\mu')\langle\nabla(s_\mu), s_\nu\rangle \in \mathbb{N}[q,t],$$

where $\iota(\mu) = \binom{\ell(\mu)}{2} + \sum_{i:\mu_i<(i-1)}(i-1-\mu_i)$. This second conjecture implies that $\nabla(s_\mu)|_{m_\nu} \in \mathbb{N}[q,t]$ for all μ and ν. Because of the signs, it is not clear whether either conjecture easily implies the other one.

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[N.L.] Mathematics Dept., College of William & Mary, Williamsburg, VA
E-mail address: nick@math.wm.edu

[G.W.] Mathematics Dept., Wake Forest University, Winston-Salem, NC
E-mail address: warrings@wfu.edu