The Sweep Map

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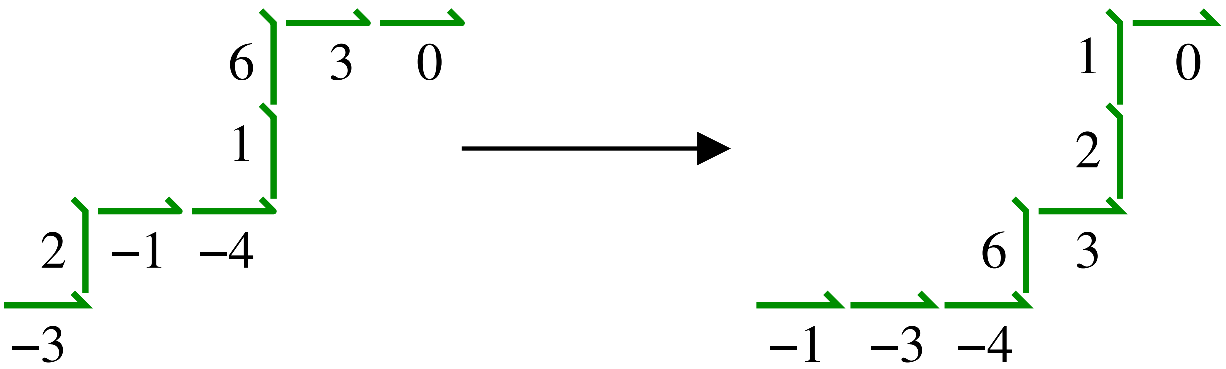

CMS Winter Meeting — Ottawa
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Nutthis.talkshell

An
injective
sorting map

Area
is the only
statistic

A typical sort

$$\boldsymbol{NENNEE}$$

Assign a weight to each letter:

$$\mathbf{wt}(\boldsymbol{N}) = \mathbf{+1}, \quad \mathbf{wt}(\boldsymbol{E}) = \mathbf{-1}.$$

Sort in decreasing order of weight.

A typical sort

$$\boldsymbol{NENNEE} \longrightarrow \boldsymbol{NNNEEE}$$

Assign a weight to each letter:

$$\mathbf{wt}(\boldsymbol{N}) = +\mathbf{1}, \quad \mathbf{wt}(\boldsymbol{E}) = -\mathbf{1}.$$

Sort in decreasing order of weight.

Define a Dyck path of order n to be a NE-lattice path from $(0,0)$ to (n,n) that stays weakly above $y=x$.

Let $w = w_1 \cdots w_n \in \mathcal{D}_n$.

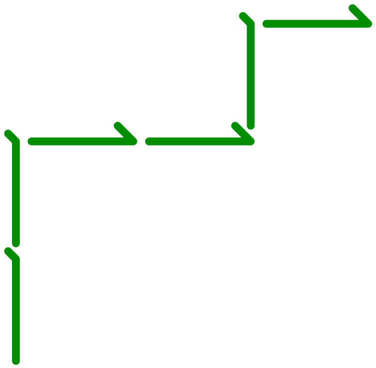
Define levels

$$\ell_i = \ell_i(w) = \begin{cases} 0, & i = 0, \\ \ell_{i-1} + \mathrm{wt}(w_i), & i > 0. \end{cases}$$

Sorting by level

Sort steps by levels: $\ldots, 2, 1, 0, -1, -2, \ldots$

Break ties from Right to Left.



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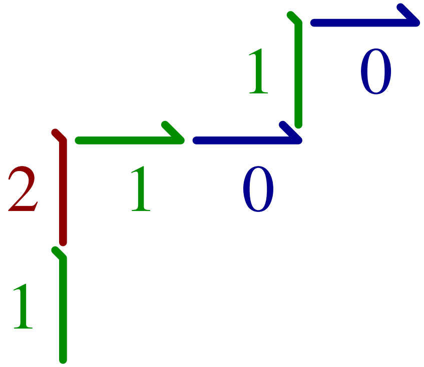

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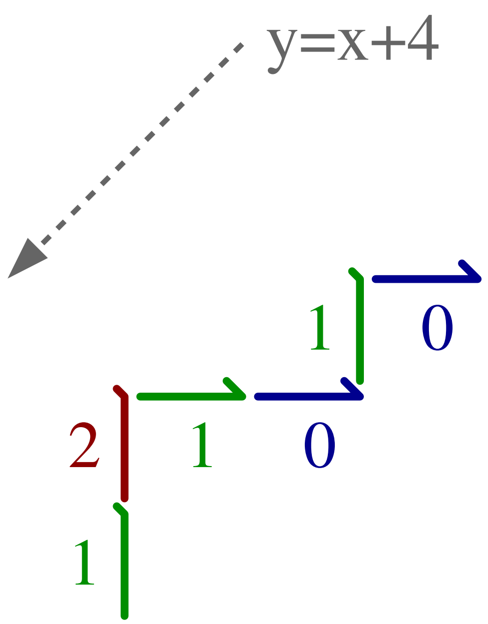

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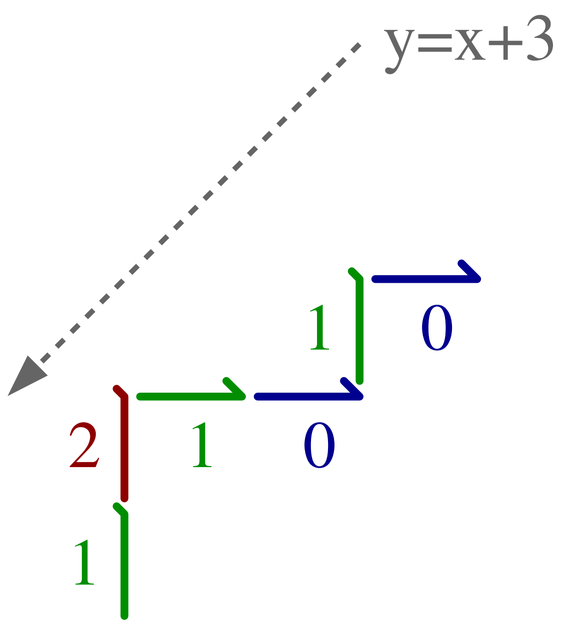

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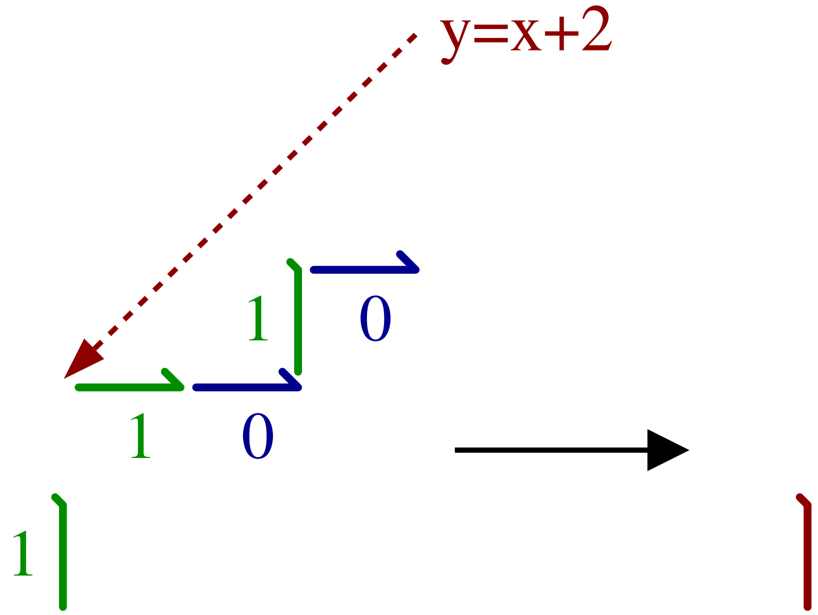

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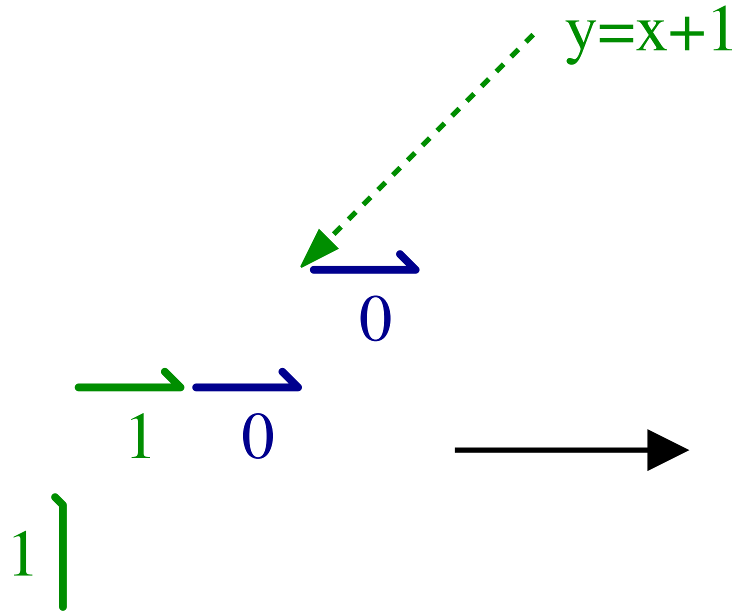

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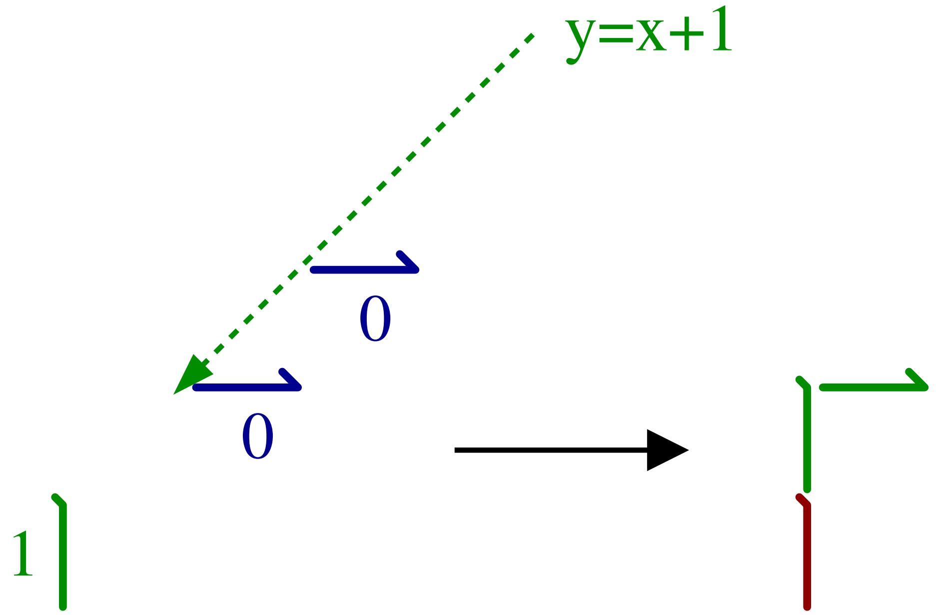

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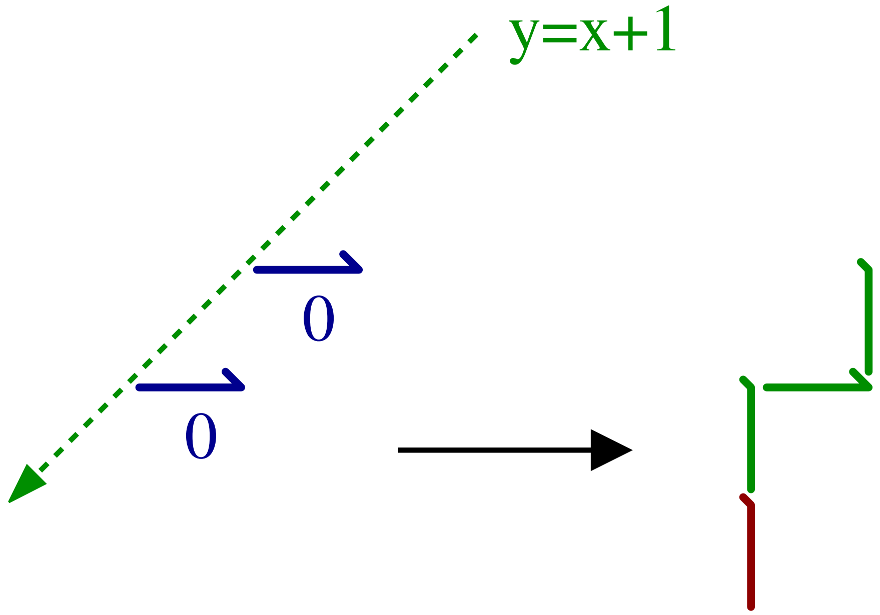

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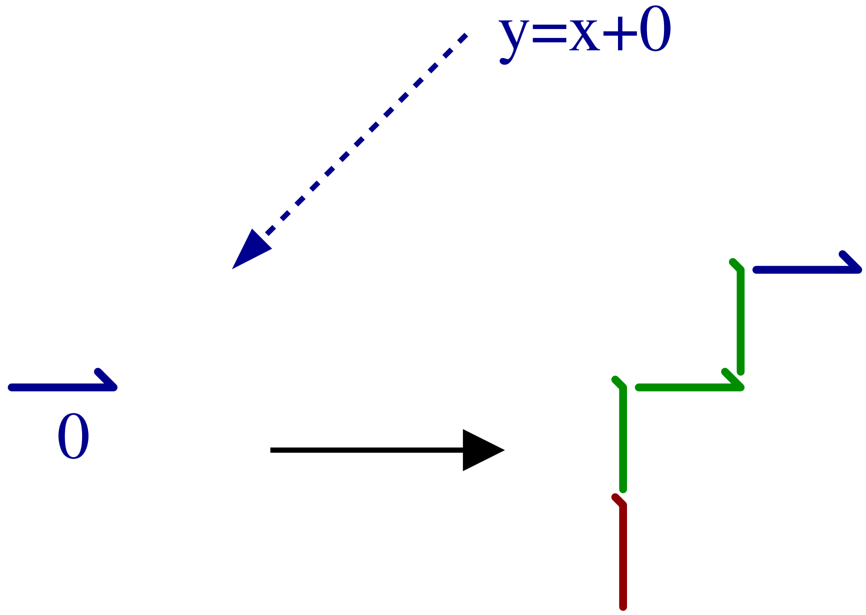

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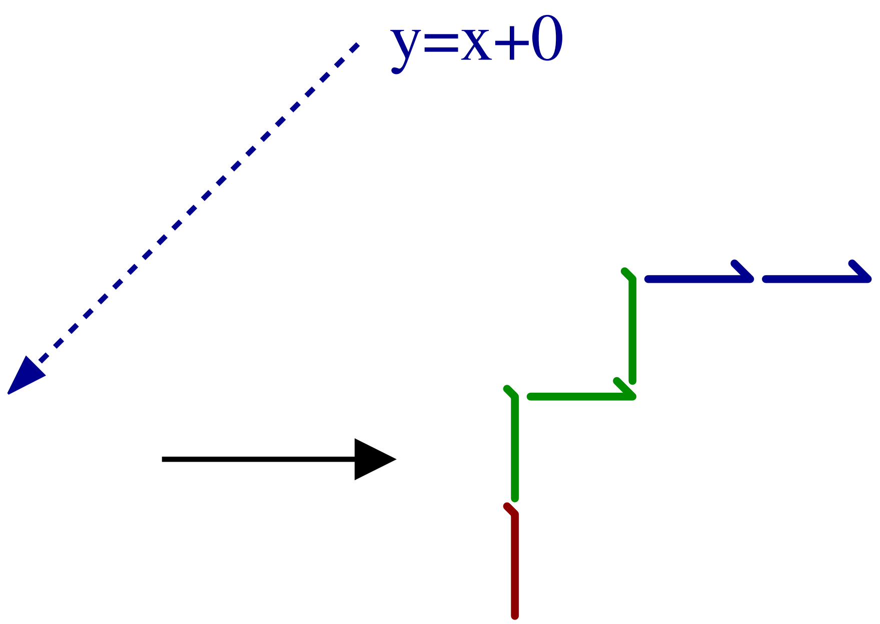

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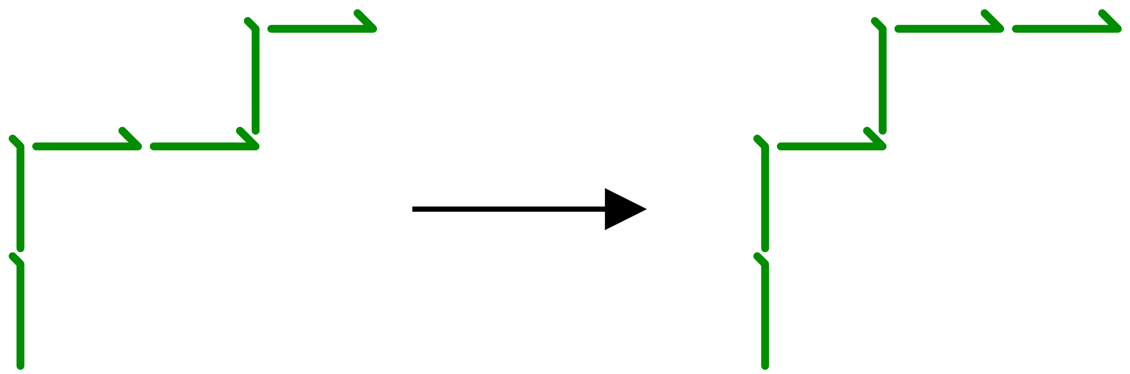
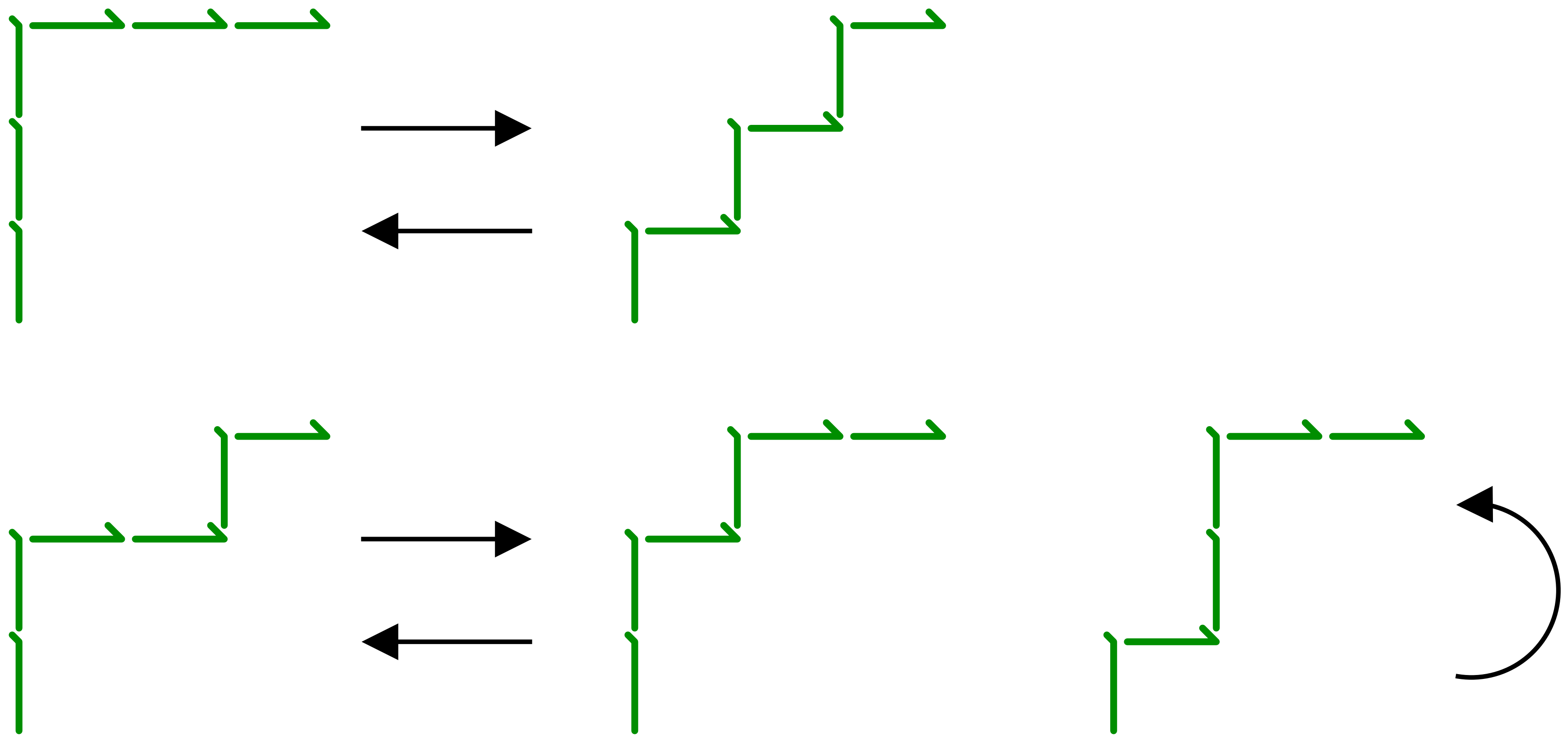


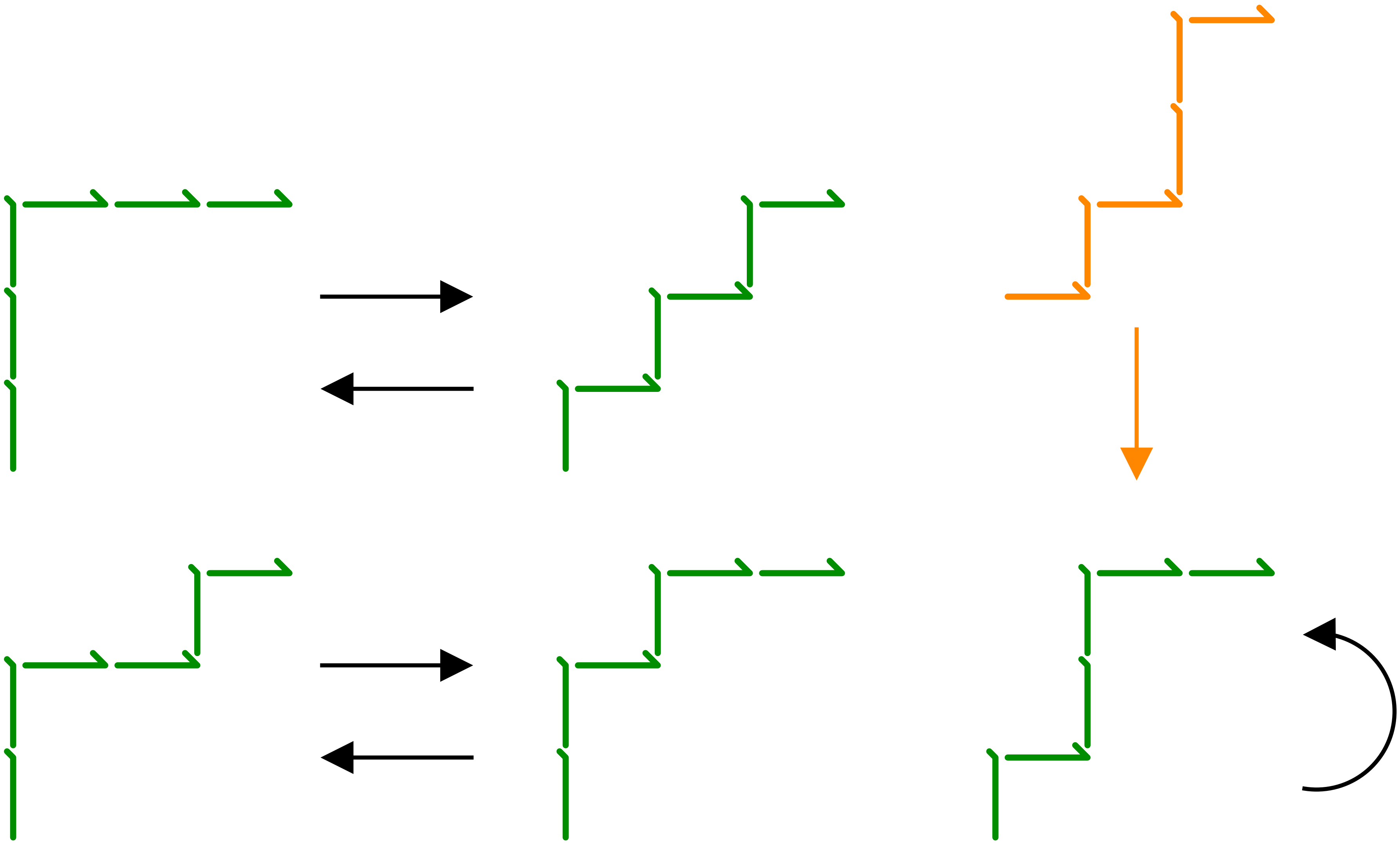
Order-3 Dyck paths



Order-3 Dyck paths

Theorem[L '03]: The map defined is a bijection on Dyck paths of order n.

Oops

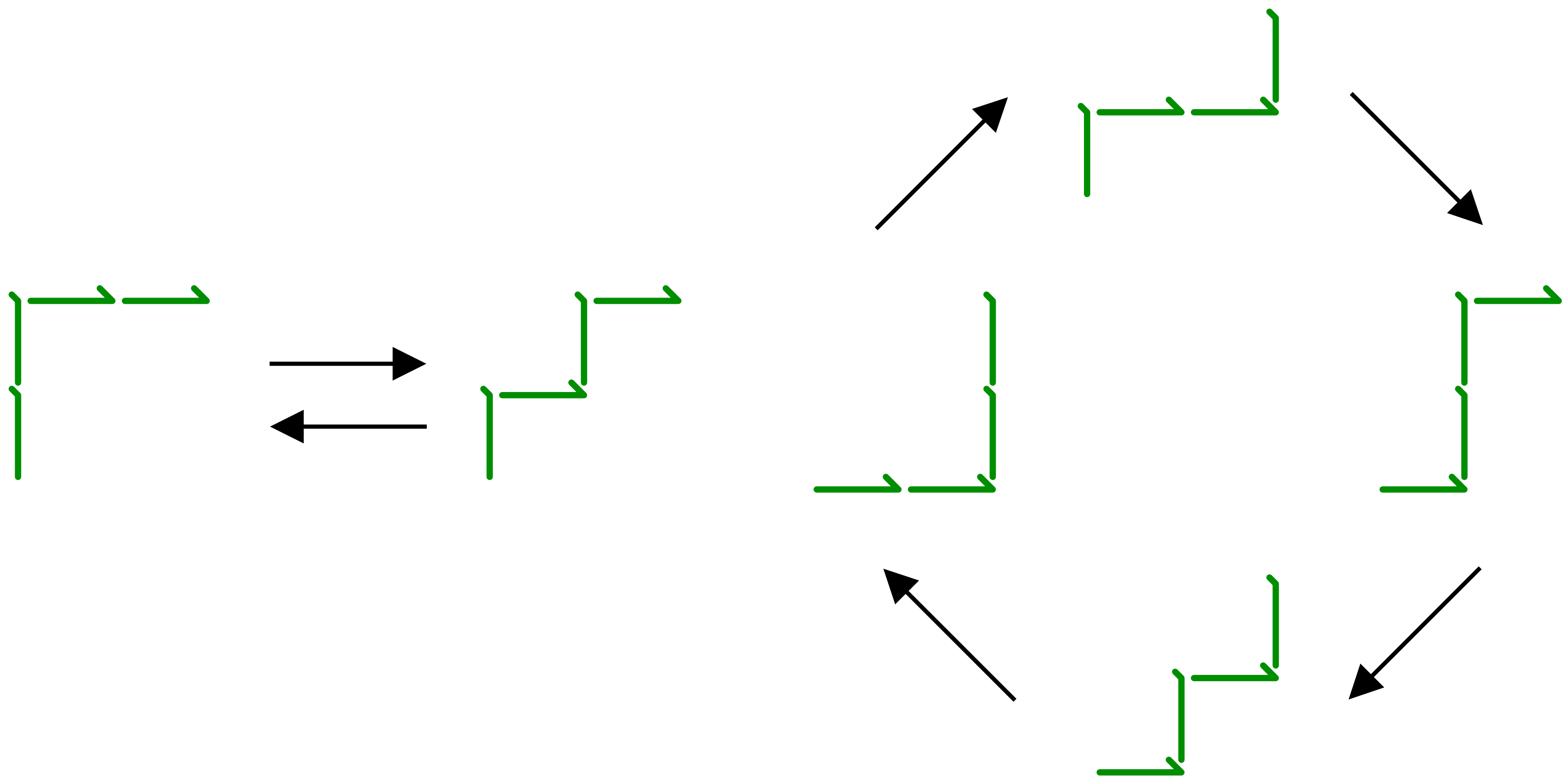


All is not lost

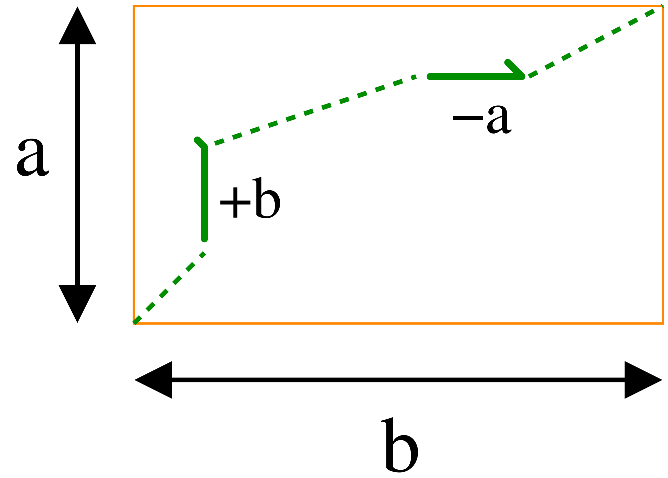
A proper sorting order for levels:

$$\mathbf{-1, -2, -3, \ldots, \ldots, 2, 1, 0.}$$

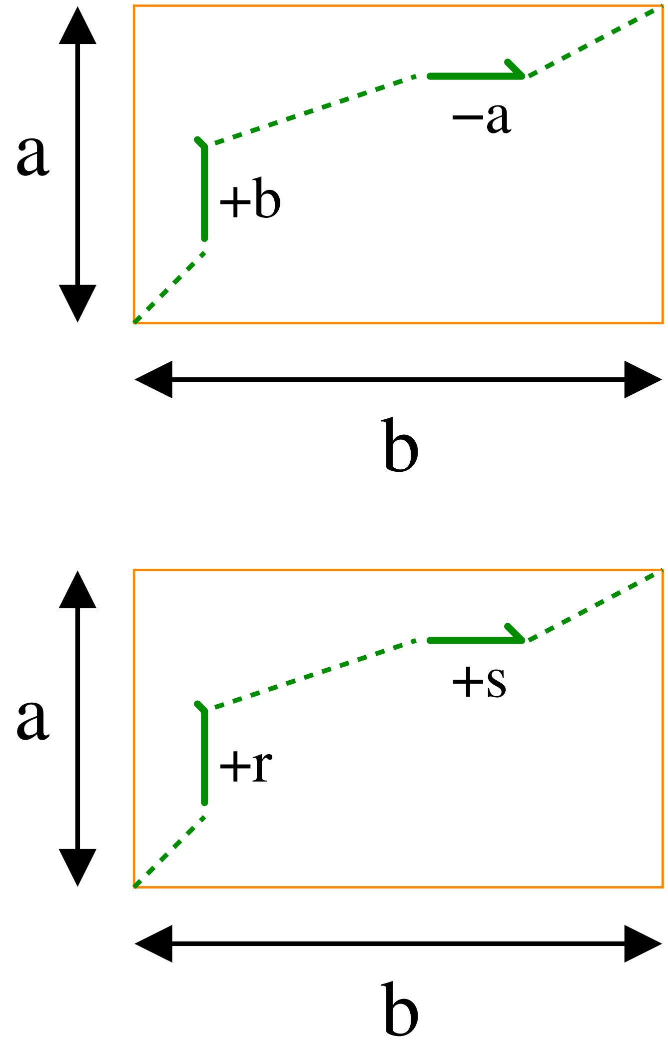
All 2 × 2 paths



Rational variations



Rational variations



The sweep map $\mathrm{sw}_{\mathrm{wt}}$

An alphabet $A = \{x_1, \ldots, x_k\}$,

A weight function $\mathrm{wt} : A \to \mathbb{Z}$,

A word $w = w_1 w_2 \cdots w_n \in A^*$

and levels

$$\ell_i = \ell_i(w) = \begin{cases} 0, & i = 0, \\ \ell_{i-1} + \mathrm{wt}(w_i), & i > 0. \end{cases}$$

Rectangular and Dyck domains

Define $\mathcal{R}(x_1^{n_1} \cdots x_k^{n_k})$ to be the set of words $w \in A^*$ consisting of n_j copies of j.

Define $\mathcal{D}_{\mathrm{wt}}(x_1^{n_1} \cdots x_k^{n_k})$ to be the set of such words for which all levels ℓ_i are nonnegative.

The Sweep Conjecture

Conjecture: For any nonnegative integers $n_1, \ldots, n_k$ and any weight-function wt,

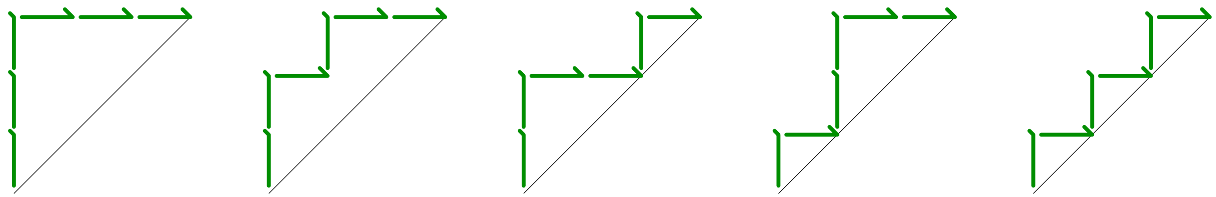
- $\mathrm{sw}_{\mathrm{wt}}$ maps $\mathcal{R}(x_1^{n_1} \cdots x_k^{n_k})$ bijectively to itself, and
- $\mathrm{sw}_{\mathrm{wt}}$ maps $\mathcal{D}_{\mathrm{wt}}(x_1^{n_1} \cdots x_k^{n_k})$ bijectively to itself.

Examples of $\mathrm{sw}_{\mathrm{wt}}$

Context	Citation	weight		
		N	E	D
Identity map		-1	-1	
Reversal map		$+1$	$+1$	
Dyck paths	L '03	$+1$	-1	
Schröder paths	EHKK '03	$+1$	-1	0
Trapezoidal paths	L '03	$+1$	$-m$	
Square paths	LW '07	$+1$	-1	
Jacobians	GM '13	$+r$	$-s$	
(a, b)-cores	AHJ '13	$+b$	$-a$	

Catalan numbers

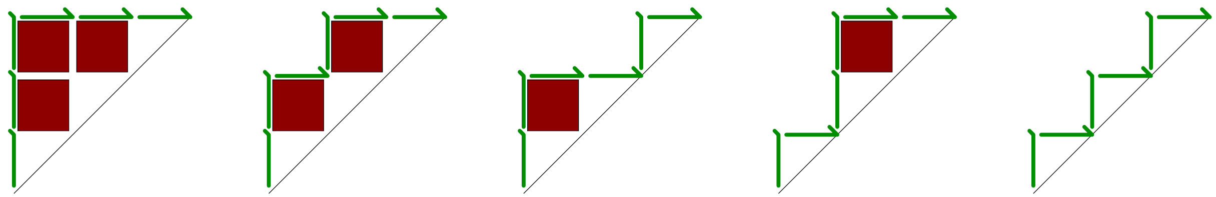
Fact: $\sum_{w \in \mathcal{D}_n} 1 = C_n = \frac{1}{n+1}\binom{2n}{n}$.



So $\sum_{w \in \mathcal{D}_3} 1 = 5 = \frac{1}{4}\binom{6}{3}$.

q-Catalan numbers

Fact: $\sum_{w \in \mathcal{D}_n} 1 = C_n = \frac{1}{n+1}\binom{2n}{n}$.



So $\sum_{w \in \mathcal{D}_3} q^{\mathsf{area}(w)} = q^3 + q^2 + 2q + 1$.

q,t-Catalan (circa 1996)

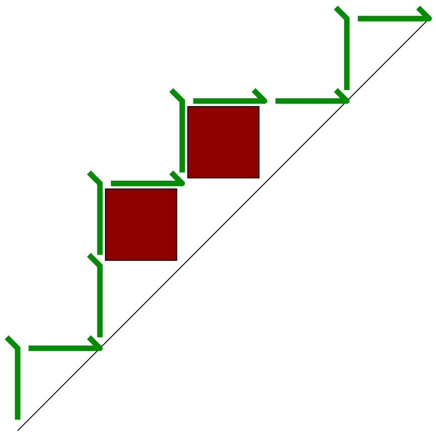
Given (G-H): Rational functions $OC_n(q,t)$

$$\text{satisfying} \qquad OC_n(q,t) = OC_n(t,q),$$
$$OC_n(1,1) = C_n,$$
$$OC_n(1,q) = OC_n(q,1) = \sum_{w \in \mathcal{D}_n} q^{\mathsf{area}(w)}.$$

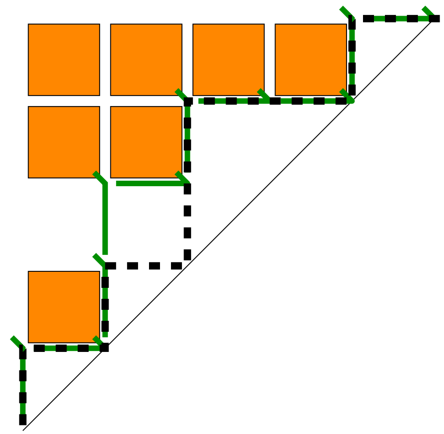
Wanted: $OC_n(q,t) = \sum_{w \in \mathcal{D}_n} q^{\mathsf{area}(w)} t^{\mathsf{tstat}(w)}.$

Statistics

Area = 2

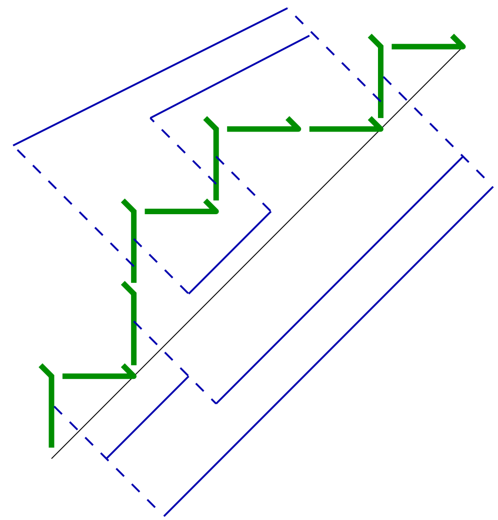


Bounce = 7



Haglund

Dinv = 6



Haiman

Theorem (G-H):

$$OC_n(q,t) = \sum_{w \in \mathcal{D}_n} q^{\mathsf{area}(w)} t^{\mathsf{bounce}(w)}.$$

Symmetry of the q, t-Catalan

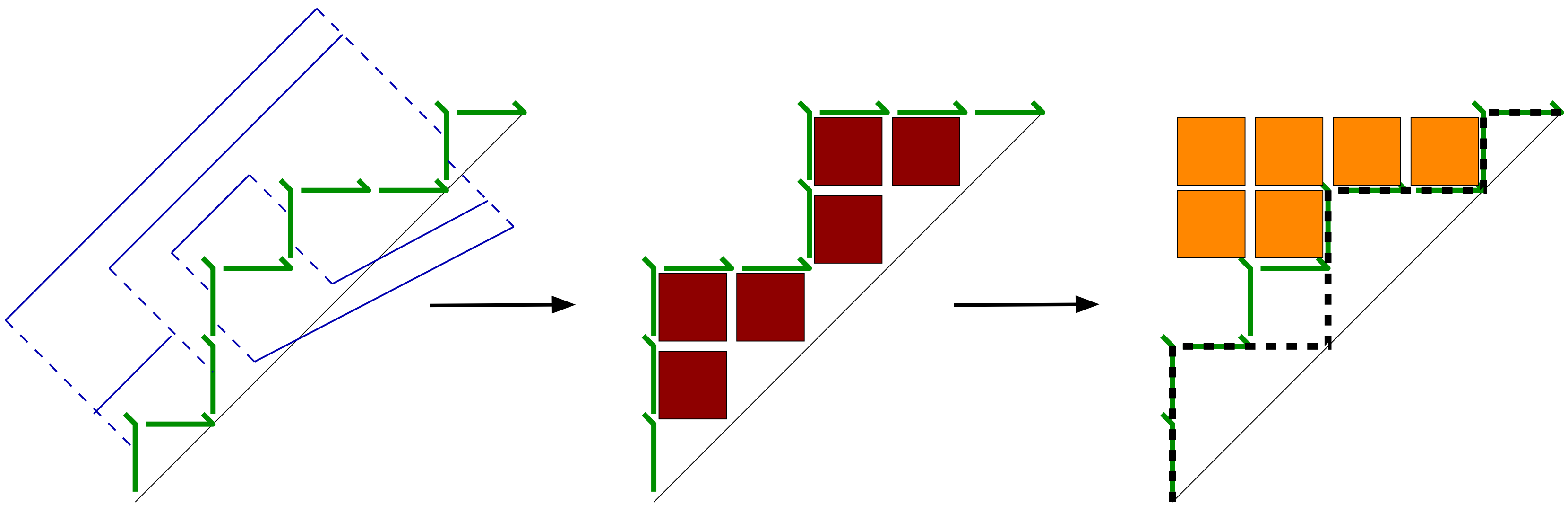
Prove combinatorially that

$$\sum_{w \in \mathcal{D}_n} q^{\mathsf{area}(w)} t^{\mathsf{dinv}(w)} = \sum_{w \in \mathcal{D}_n} q^{\mathsf{dinv}(w)} t^{\mathsf{area}(w)},$$

Or, equivalently, that

$$\sum_{w \in \mathcal{D}_n} q^{\mathsf{area}(w)} t^{\mathsf{bounce}(w)} = \sum_{w \in \mathcal{D}_n} q^{\mathsf{bounce}(w)} t^{\mathsf{area}(w)}.$$

Sweeping up statistics



Dinv	6	3	7
Area	2	6	3
Bounce	7	2	6

Aaargh

Symmetry of the q, t-Catalan

Prove combinatorially that

$$\sum_{w \in \mathcal{D}_n} q^{\text{area}(w)} t^{\text{area}(\mathrm{sw}(w))} = \sum_{w \in \mathcal{D}_n} q^{\text{area}(\mathrm{sw}(w))} t^{\text{area}(w)},$$

Or, equivalently, that

$$\sum_{w \in \mathcal{D}_n} q^{\text{area}(w)} t^{\text{area}(\mathrm{sw}^{-1}(w))} = \sum_{w \in \mathcal{D}_n} q^{\text{area}(\mathrm{sw}^{-1}(w))} t^{\text{area}(w)}.$$

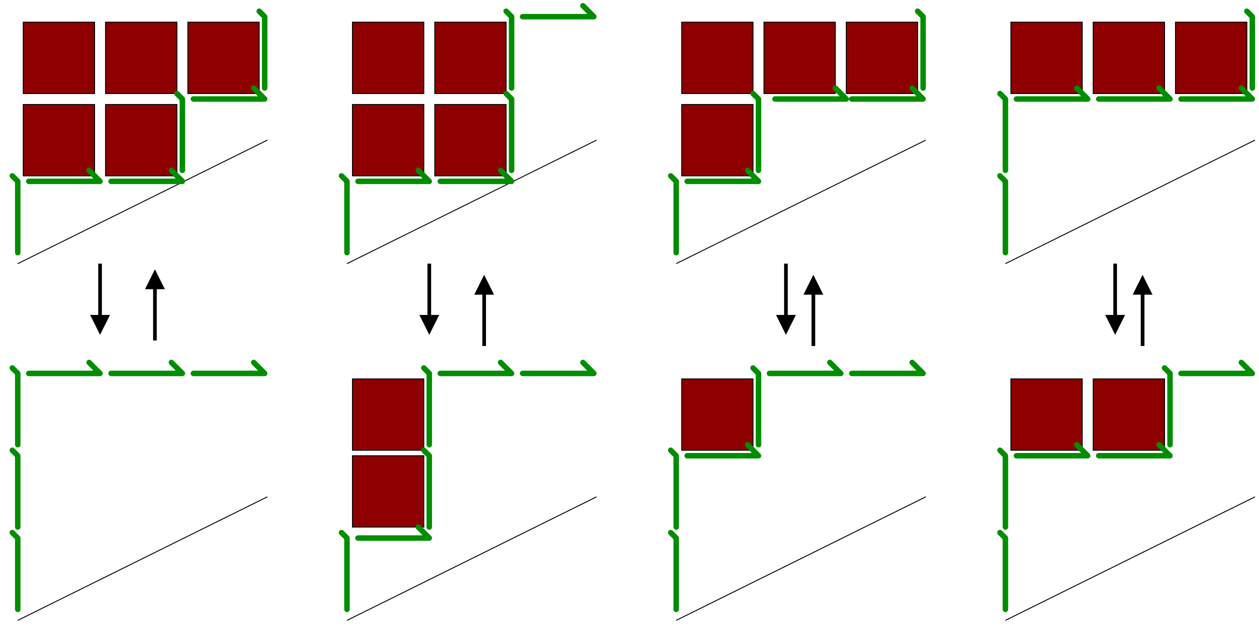
Slope-$(-s/r)$ q,t-Catalan

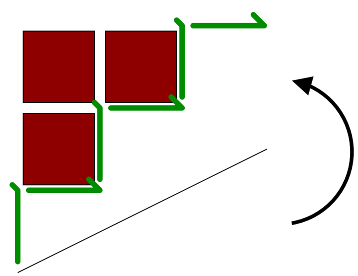
For $r,s \in \mathbb{Z}$ and $a,b \geq 0$,
define

$$C_{r,s,a,b}(q,t) = \sum_{w \in \mathcal{D}_{r,s}(N^a E^b)} q^{\text{area}(w)} t^{\text{area}(\mathrm{sw}_{r,s}(w))}.$$

Conjecture: $C_{r,s,a,b}(q,t) = C_{r,s,a,b}(t,q)$.

Example: $a = b = 3, r = 2, s = -1$





	0	1	2	3	4	5
0						1
1					1	
2				1	1	
3			1	1		
4		1	1			
5	1					