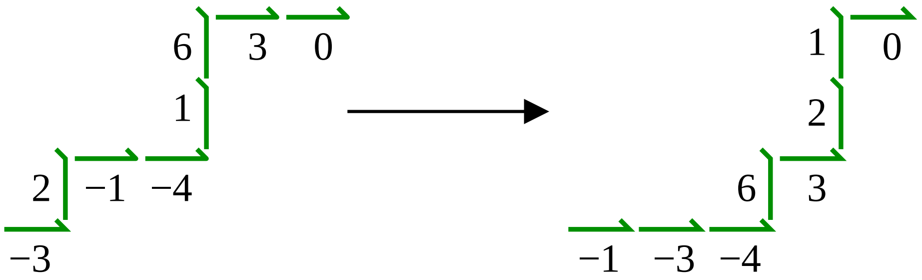
Rational Catalan numbers and The Sweep Map

Drew Armstrong — University of Miami

Nick Loehr — Virginia Tech & US Naval Academy

Greg Warrington — University of Vermont



UVM-St. Michael's Combinatorics Seminar

November 6, 2014

Part I

An

injective

sorting map

A typical sort

$$\boldsymbol{NENNEE}$$

Assign a weight to each letter:

$$\mathbf{wt}(\boldsymbol{N}) = +\mathbf{1}, \quad \mathbf{wt}(\boldsymbol{E}) = -\mathbf{1}.$$

Sort in decreasing order of weight.

A typical sort

$$\boldsymbol{NENNEE} \longrightarrow \boldsymbol{NNNEEE}$$

Assign a weight to each letter:

$$\mathbf{wt}(\boldsymbol{N}) = +\mathbf{1}, \quad \mathbf{wt}(\boldsymbol{E}) = -\mathbf{1}.$$

Sort in decreasing order of weight.

Define a Dyck path of order n to be a NE-lattice path from $(0, 0)$ to (n, n) that stays weakly above $y = x$.

Let $w = w_1 \cdots w_n \in \mathcal{D}_n$.

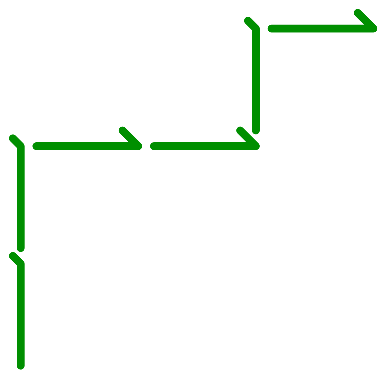
Define levels

$$\ell_i = \ell_i(w) = \begin{cases} 0, & i = 0, \\ \ell_{i-1} + \mathrm{wt}(w_i), & i > 0. \end{cases}$$

Sorting by level

Sort steps by levels: $\ldots, \mathbf{2}, \mathbf{1}, \mathbf{0}, \mathbf{-1}, \mathbf{-2}, \ldots$

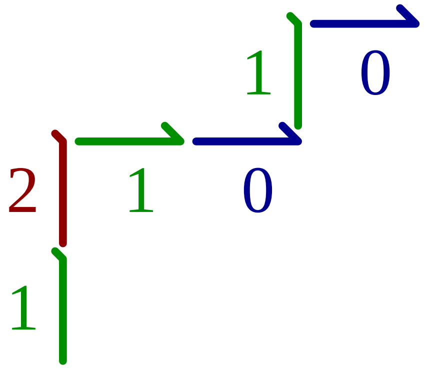
Break ties from Right to Left.



Sorting by level

Sort steps by levels: $\ldots, \mathbf{2}, \mathbf{1}, \mathbf{0}, \mathbf{-1}, \mathbf{-2}, \ldots$

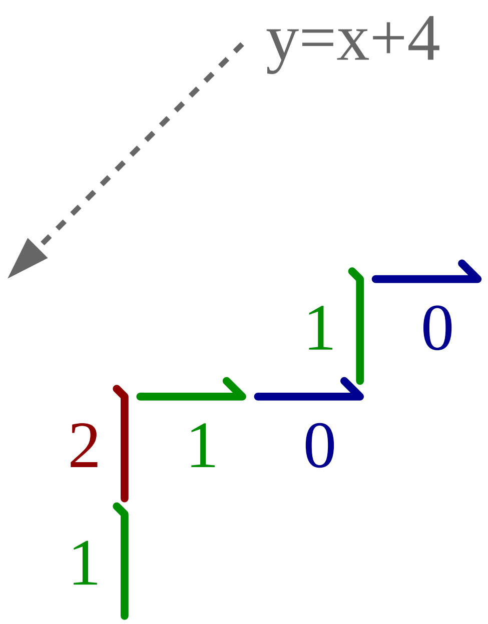
Break ties from Right to Left.



Sorting by level

Sort steps by levels: $\ldots, \mathbf{2}, \mathbf{1}, \mathbf{0}, \mathbf{-1}, \mathbf{-2}, \ldots$

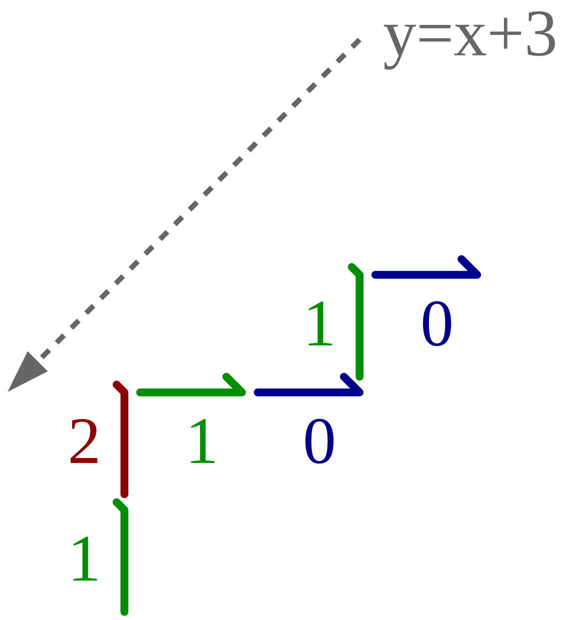
Break ties from Right to Left.



Sorting by level

Sort steps by levels: $\ldots, \mathbf{2}, \mathbf{1}, \mathbf{0}, \mathbf{-1}, \mathbf{-2}, \ldots$

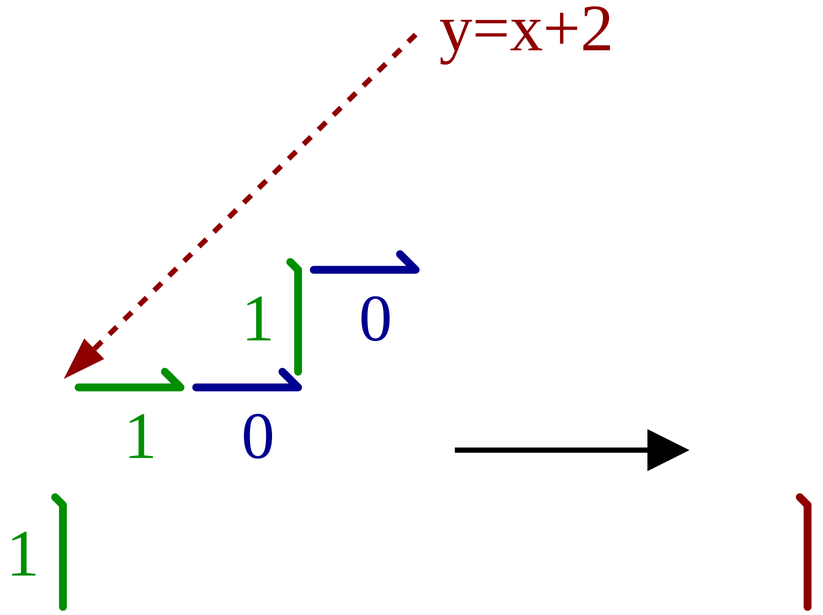
Break ties from Right to Left.



Sorting by level

Sort steps by levels: $\ldots, \mathbf{2}, \mathbf{1}, \mathbf{0}, \mathbf{-1}, \mathbf{-2}, \ldots$

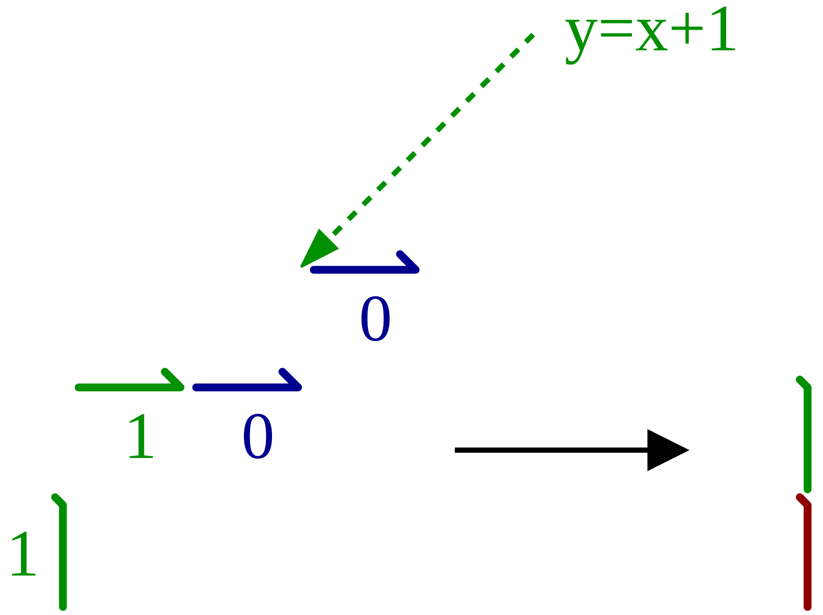
Break ties from Right to Left.



Sorting by level

Sort steps by levels: $\ldots, \mathbf{2}, \mathbf{1}, \mathbf{0}, \mathbf{-1}, \mathbf{-2}, \ldots$

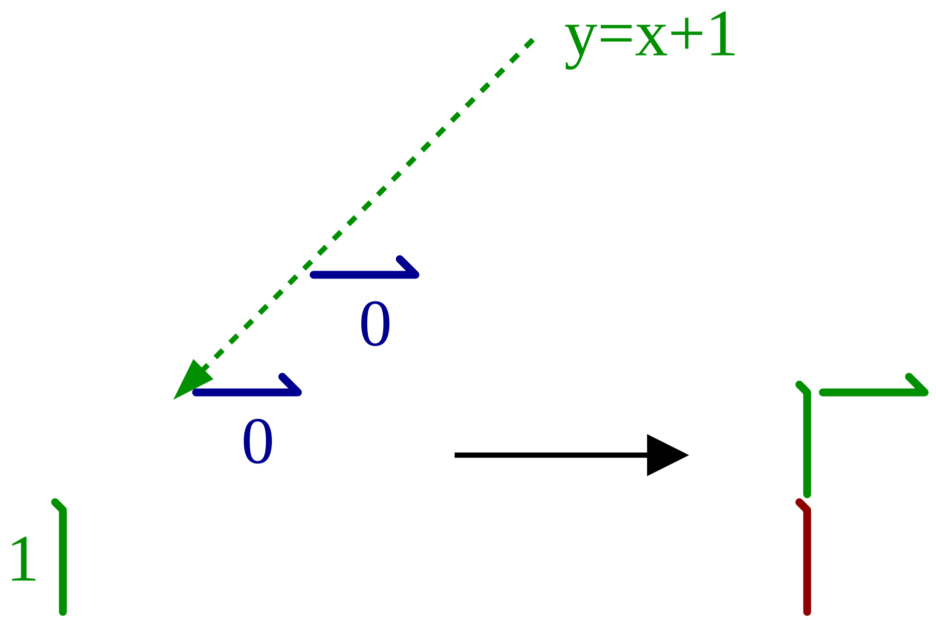
Break ties from Right to Left.



Sorting by level

Sort steps by levels: $\ldots, \mathbf{2}, \mathbf{1}, \mathbf{0}, \mathbf{-1}, \mathbf{-2}, \ldots$

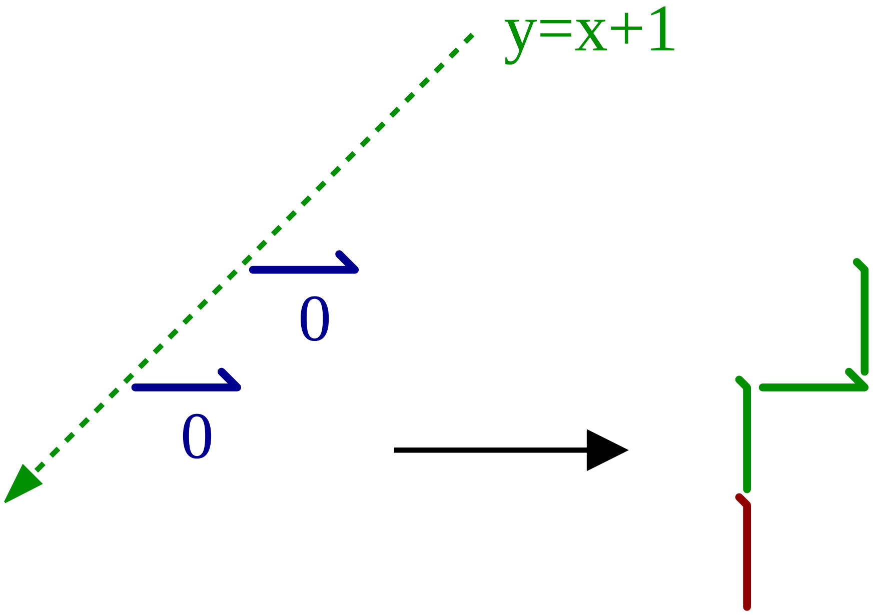
Break ties from Right to Left.



Sorting by level

Sort steps by levels: $\mathbf{\ldots, 2, 1, 0, -1, -2, \ldots}$

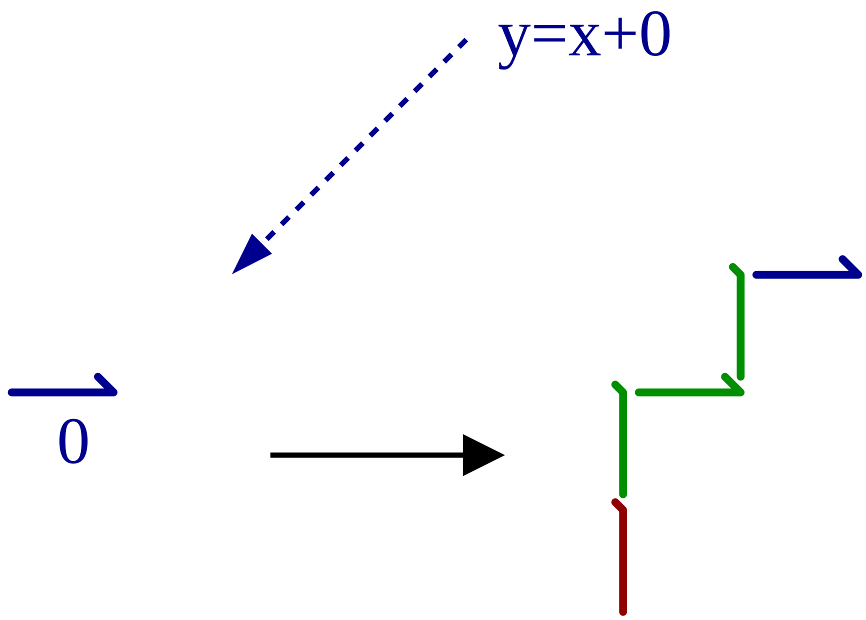
Break ties from Right to Left.



Sorting by level

Sort steps by levels: $\ldots, \mathbf{2}, \mathbf{1}, \mathbf{0}, \mathbf{-1}, \mathbf{-2}, \ldots$

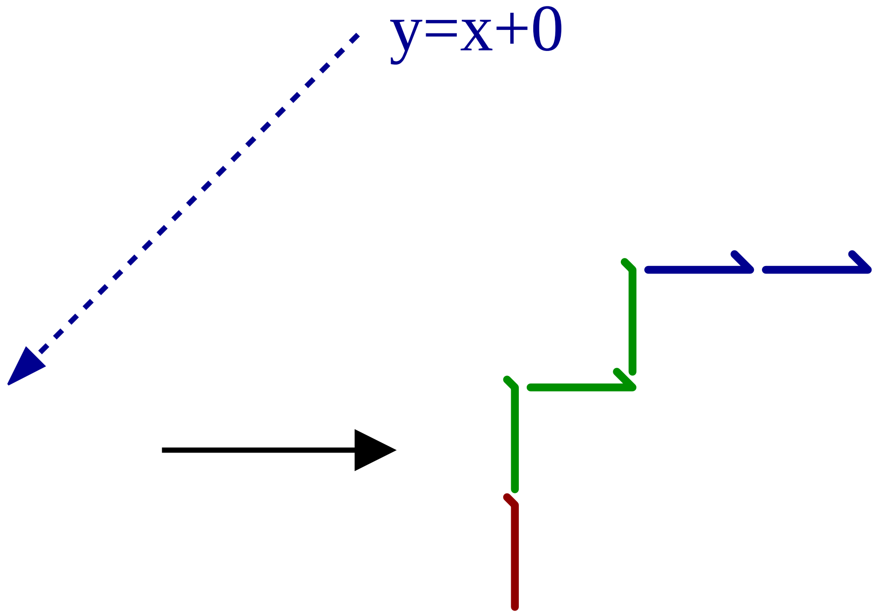
Break ties from Right to Left.



Sorting by level

Sort steps by levels: $\ldots, \mathbf{2}, \mathbf{1}, \mathbf{0}, \mathbf{-1}, \mathbf{-2}, \ldots$

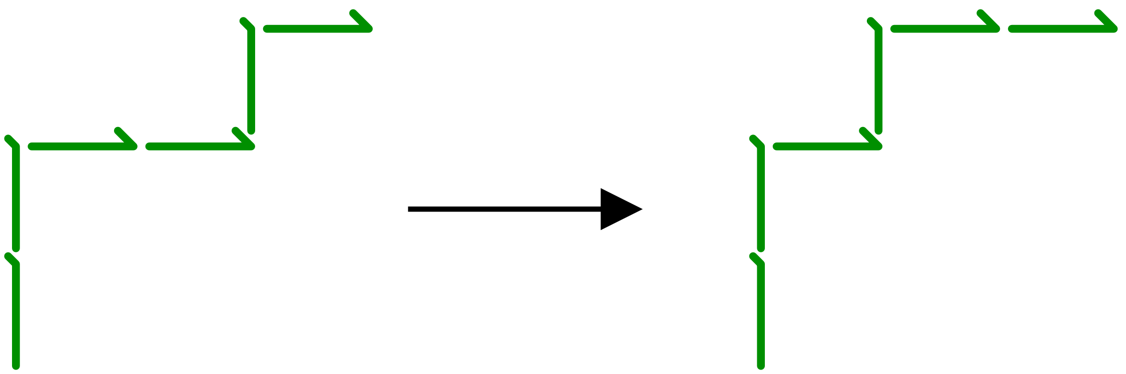
Break ties from Right to Left.



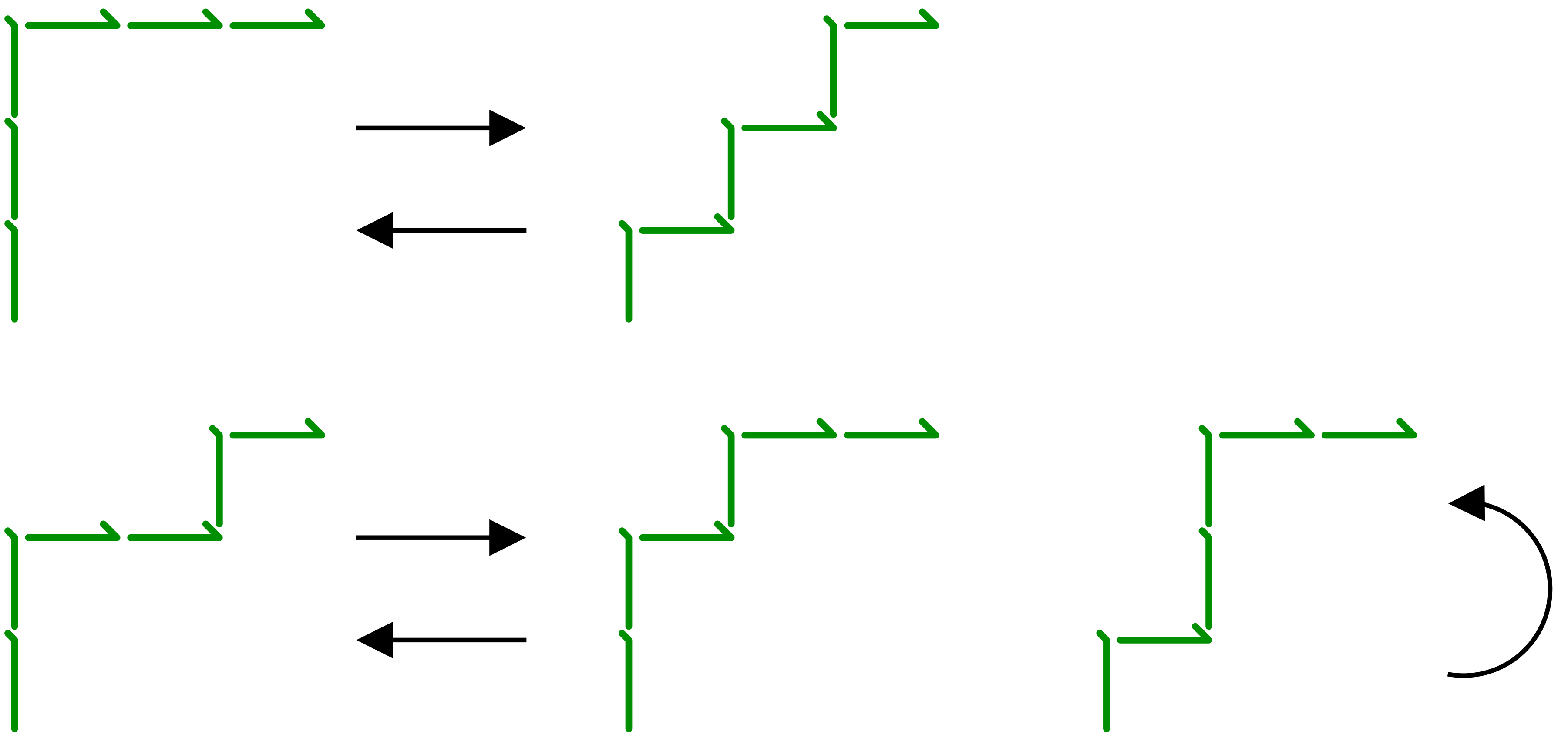
Sorting by level

Sort steps by levels: $\ldots, \mathbf{2}, \mathbf{1}, \mathbf{0}, \mathbf{-1}, \mathbf{-2}, \ldots$

Break ties from Right to Left.



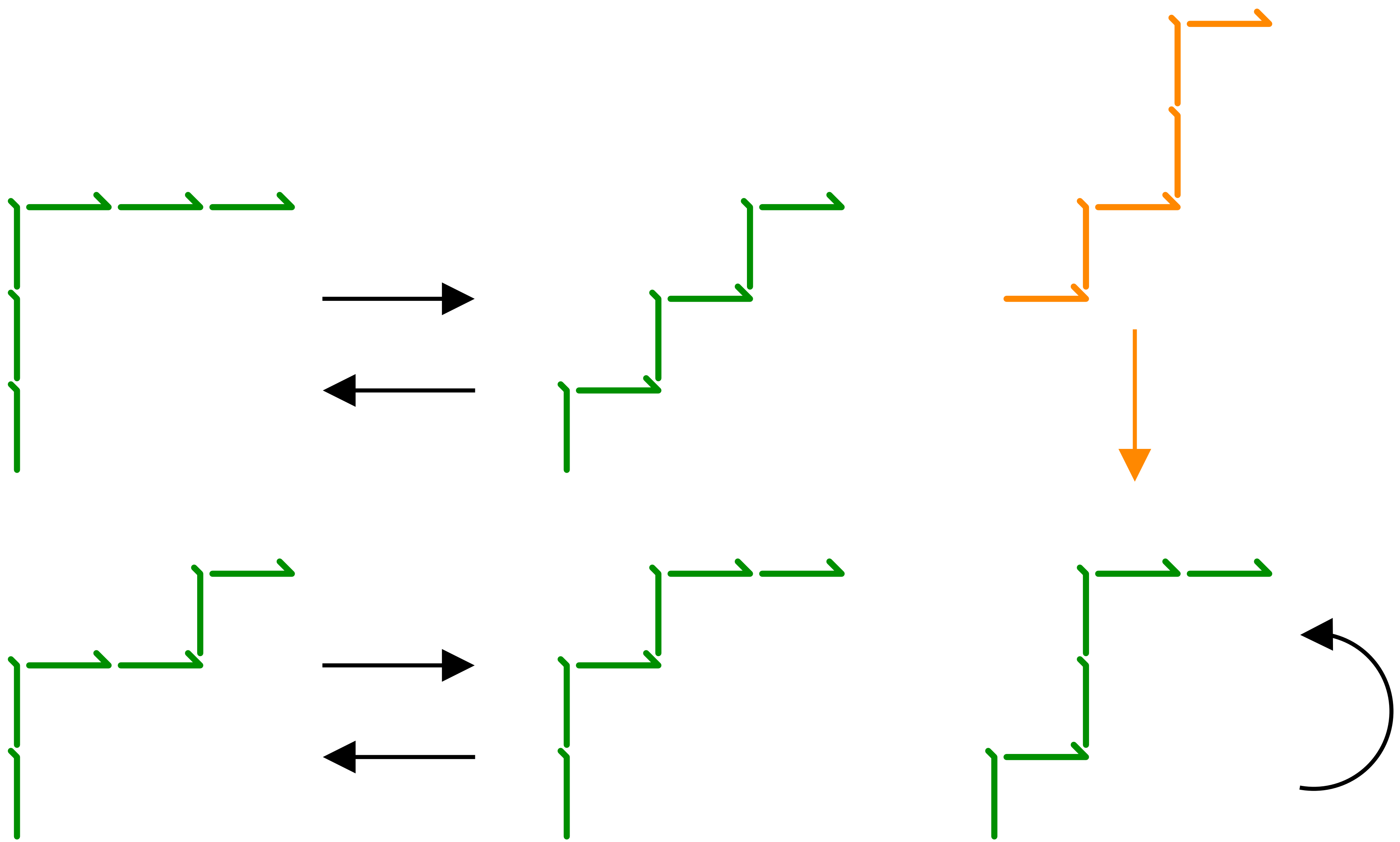
Order-3 Dyck paths



Order-3 Dyck paths

Theorem[L '03]: The map defined is a bijection on Dyck paths of order n.

Oops

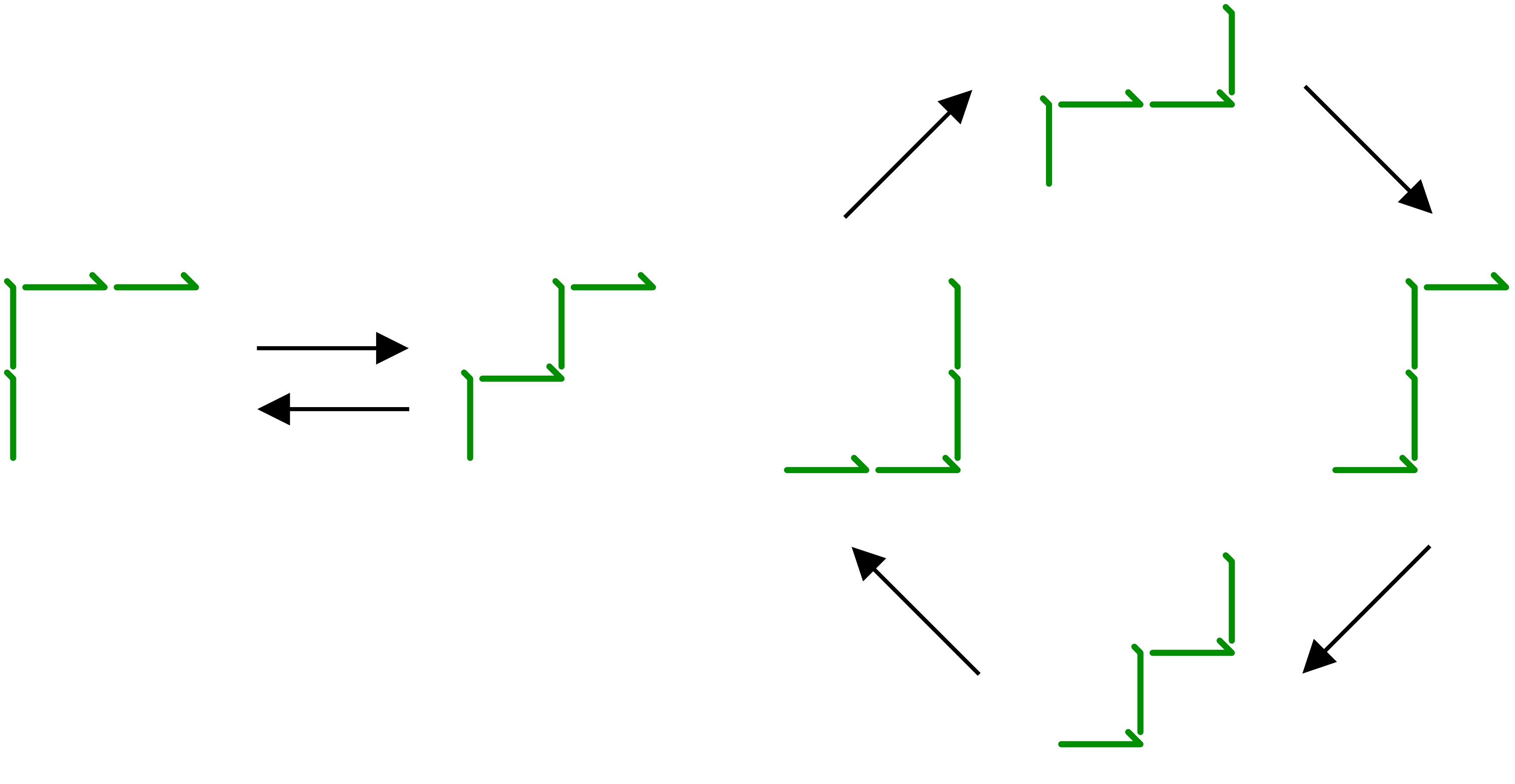


All is not lost

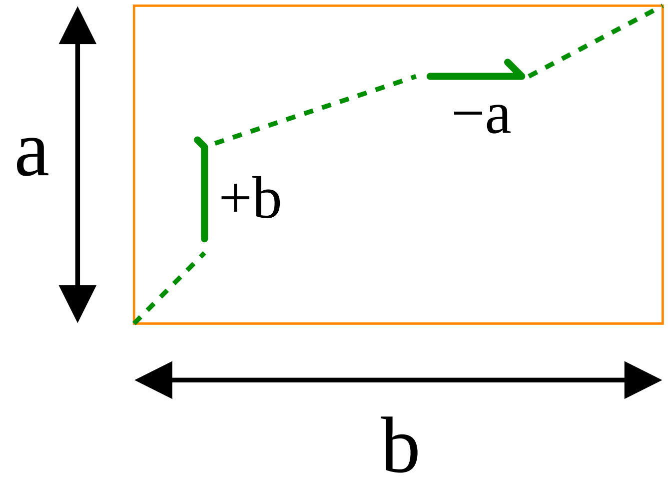
A proper sorting order for levels:

$$-1, -2, -3, \ldots, \ldots, 2, 1, 0.$$

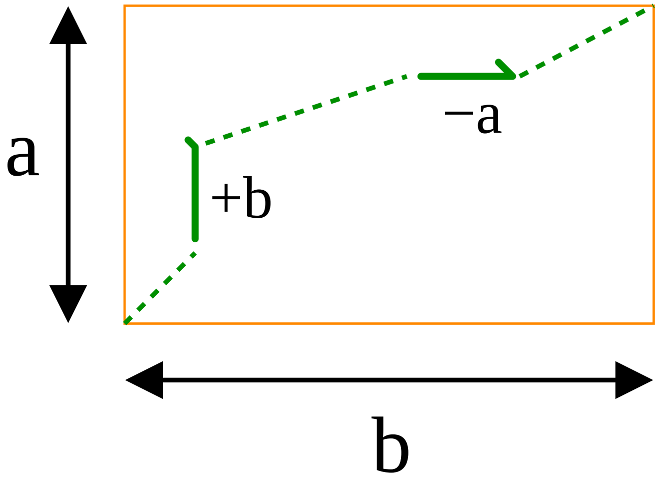
All 2 × 2 paths

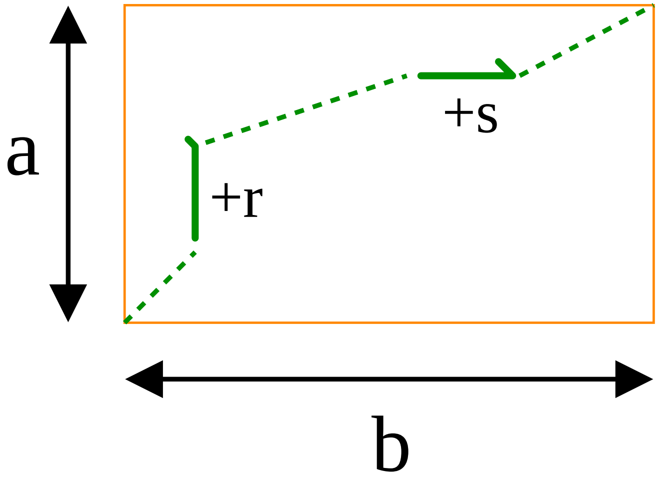


Rational variations



Rational variations





The sweep map $\mathrm{sw}_{\mathrm{wt}}$

An alphabet $A = \{x_1, \ldots, x_k\}$,

A weight function $\mathbf{wt} : A \to \mathbb{Z}$,

A word $w = w_1 w_2 \cdots w_n \in A^*$

and levels

$$\ell_i = \ell_i(w) = \begin{cases} 0, & i = 0, \\ \ell_{i-1} + \mathrm{wt}(w_i), & i > 0. \end{cases}$$

Rectangular and Dyck domains

Define $\mathcal{R}(x_1^{n_1} \cdots x_k^{n_k})$ to be the set of words $w \in A^*$ consisting of n_j copies of j.

Define $\mathcal{D}_{\mathrm{wt}}(x_1^{n_1} \cdots x_k^{n_k})$ to be the set of such words for which all levels ℓ_i are nonnegative.

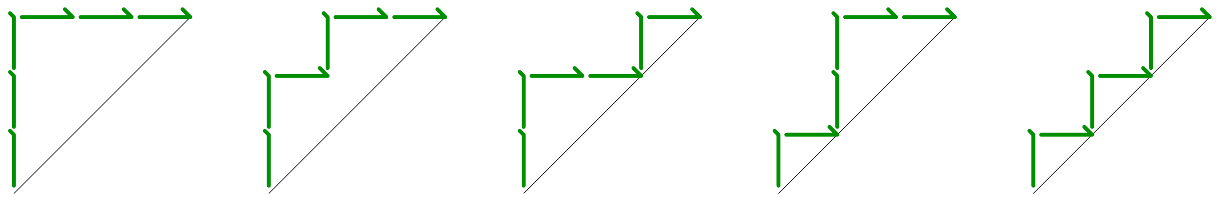
The Sweep Conjecture

Conjecture: For any nonnegative integers $n_1, \ldots, n_k$ and any weight-function $\mathbf{wt}$,

- $\mathbf{sw}_{\mathbf{wt}}$ maps $\mathcal{R}(x_1^{n_1} \cdots x_k^{n_k})$ bijectively to itself, and
- $\mathbf{sw}_{\mathbf{wt}}$ maps $\mathcal{D}_{\mathbf{wt}}(x_1^{n_1} \cdots x_k^{n_k})$ bijectively to itself.

Catalan numbers

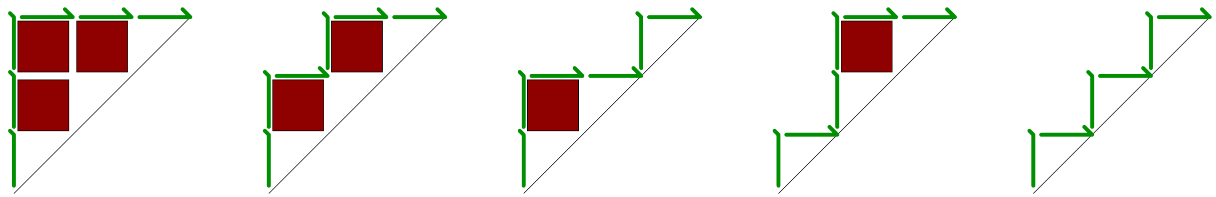
Fact: $\sum_{w\in\mathcal{D}_n} 1 = C_n = \frac{1}{n+1}\binom{2n}{n}.$



So $\sum_{w\in\mathcal{D}_3} 1 = 5 = \frac{1}{4}\binom{6}{3}.$

q-Catalan numbers

Fact: $\sum_{w \in \mathcal{D}_n} 1 = C_n = \frac{1}{n+1}\binom{2n}{n}$.



So $\sum_{w \in \mathcal{D}_3} q^{\mathsf{area}(w)} = q^3 + q^2 + 2q + 1$.

q,t-Catalan (circa 1996)

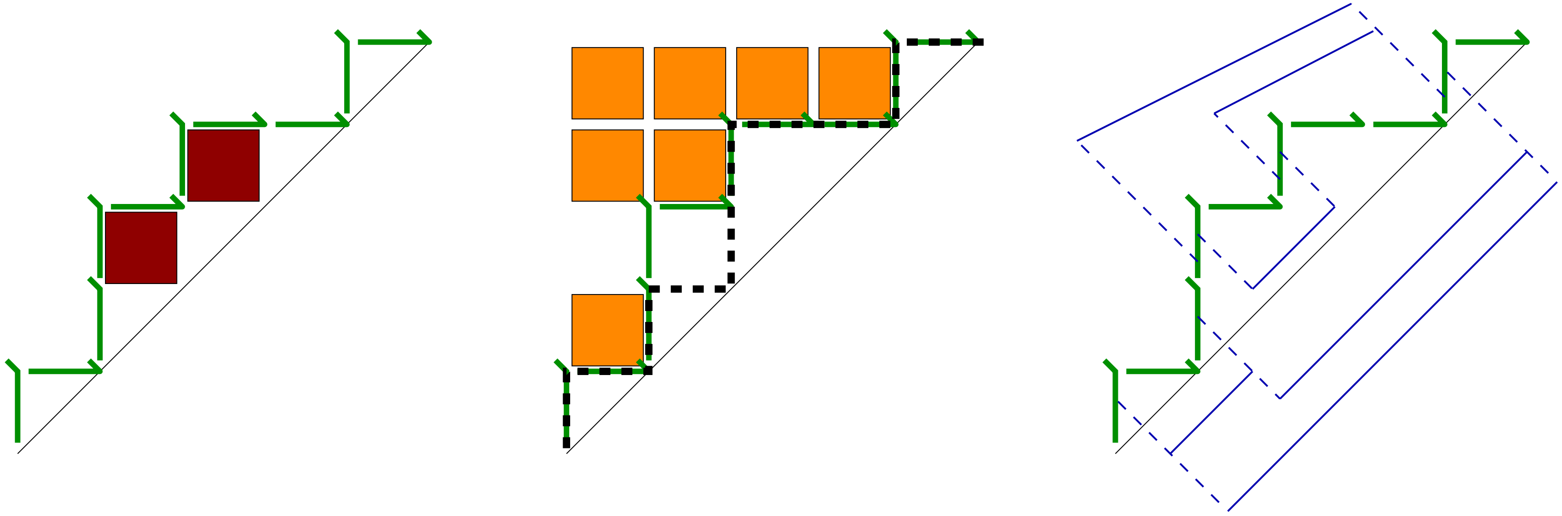
Given (G-H): Rational functions $OC_n(q,t)$

satisfying
$$OC_n(q,t) = OC_n(t,q),$$
$$OC_n(1,1) = C_n,$$
$$OC_n(1,q) = OC_n(q,1) = \sum_{w \in \mathcal{D}_n} q^{\mathsf{area}(w)}.$$

Wanted: $OC_n(q,t) = \sum_{w \in \mathcal{D}_n} q^{\mathsf{area}(w)} t^{\mathsf{tstat}(w)}.$

Statistics

Area = 2 | Bounce = 7 | Dinv = 6



Haglund | Haiman

Theorem (G-H):

$$OC_n(q,t) = \sum_{w \in \mathcal{D}_n} q^{\mathsf{area}(w)} t^{\mathsf{bounce}(w)}.$$

Symmetry of the q, t-Catalan

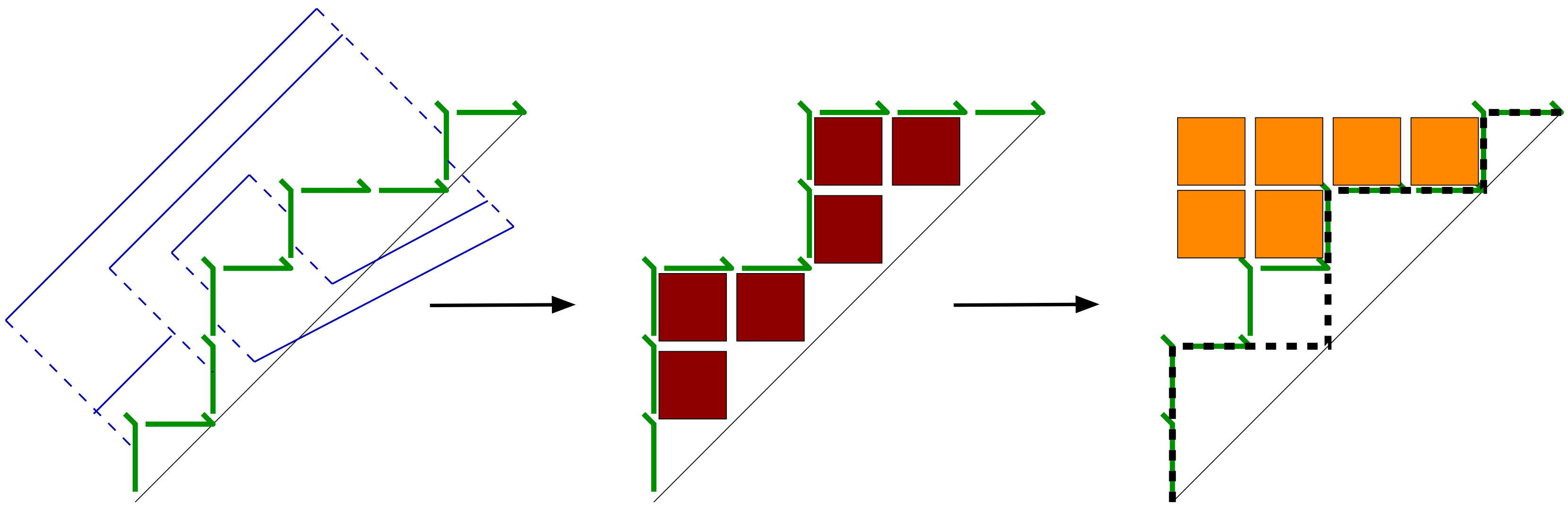
Prove combinatorially that

$$\sum_{w \in \mathcal{D}_n} q^{\mathsf{area}(w)} t^{\mathsf{dinv}(w)} = \sum_{w \in \mathcal{D}_n} q^{\mathsf{dinv}(w)} t^{\mathsf{area}(w)},$$

Or, equivalently, that

$$\sum_{w \in \mathcal{D}_n} q^{\mathsf{area}(w)} t^{\mathsf{bounce}(w)} = \sum_{w \in \mathcal{D}_n} q^{\mathsf{bounce}(w)} t^{\mathsf{area}(w)}.$$

Sweeping up statistics



Dinv	6	3	7
Area	2	6	3
Bounce	7	2	6

Aaargh

Symmetry of the q, t-Catalan

Prove combinatorially that

$$\sum_{w \in \mathcal{D}_n} q^{\text{area}(w)} t^{\text{area}(\mathrm{sw}(w))} = \sum_{w \in \mathcal{D}_n} q^{\text{area}(\mathrm{sw}(w))} t^{\text{area}(w)},$$

Or, equivalently, that

$$\sum_{w \in \mathcal{D}_n} q^{\text{area}(w)} t^{\text{area}(\mathrm{sw}^{-1}(w))} = \sum_{w \in \mathcal{D}_n} q^{\text{area}(\mathrm{sw}^{-1}(w))} t^{\text{area}(w)}.$$

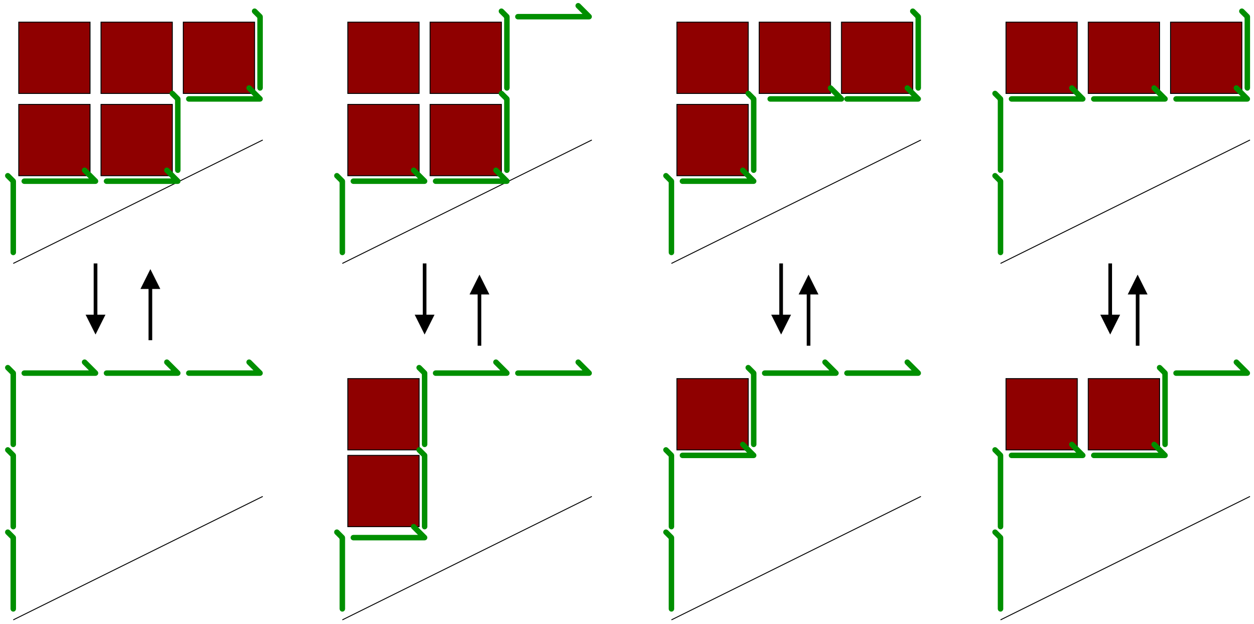
Slope-$(-s/r)$ q,t-Catalan

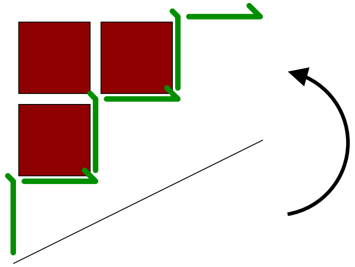
For $r, s \in \mathbb{Z}$ and $a, b \geq 0$,
define

$$C_{r,s,a,b}(q,t) = \sum_{w \in \mathcal{D}_{r,s}(N^a E^b)} q^{\mathsf{area}(w)} t^{\mathsf{area}(\mathrm{sw}_{r,s}(w))}.$$

Conjecture: $C_{r,s,a,b}(q,t) = C_{r,s,a,b}(t,q)$.

Example: $a = b = 3, r = 2, s = -1$





	0	1	2	3	4	5
0						1
1					1	
2				1	1	
3			1	1		
4		1	1			
5	1					

Part II

Find combinatorial interpretations for rational q-Catalan numbers and q-binomials.

q-binomials

$$[n]_q = 1 + q + q^2 + \cdots + q^{n-1},$$

$$[n]!_q = [n]_q[n-1]_q \cdots [2]_q[1]_q,$$

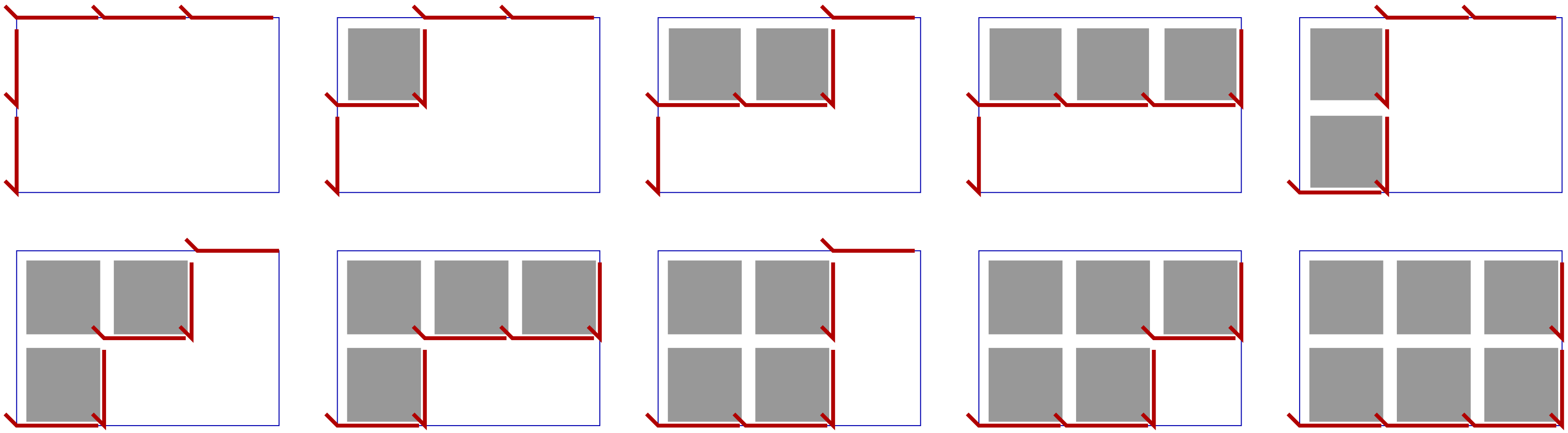
$$\begin{bmatrix} a+b \\ a, b \end{bmatrix}_q = \frac{[a+b]!_q}{[a]!_q[b]!_q}.$$

q-binomials

Theorem:

$$\begin{bmatrix} a+b \\ a,b \end{bmatrix}_q = \sum_{\mu \in \mathcal{R}^{\mathrm{ptn}}(a,b)} q^{|\mu|}.$$

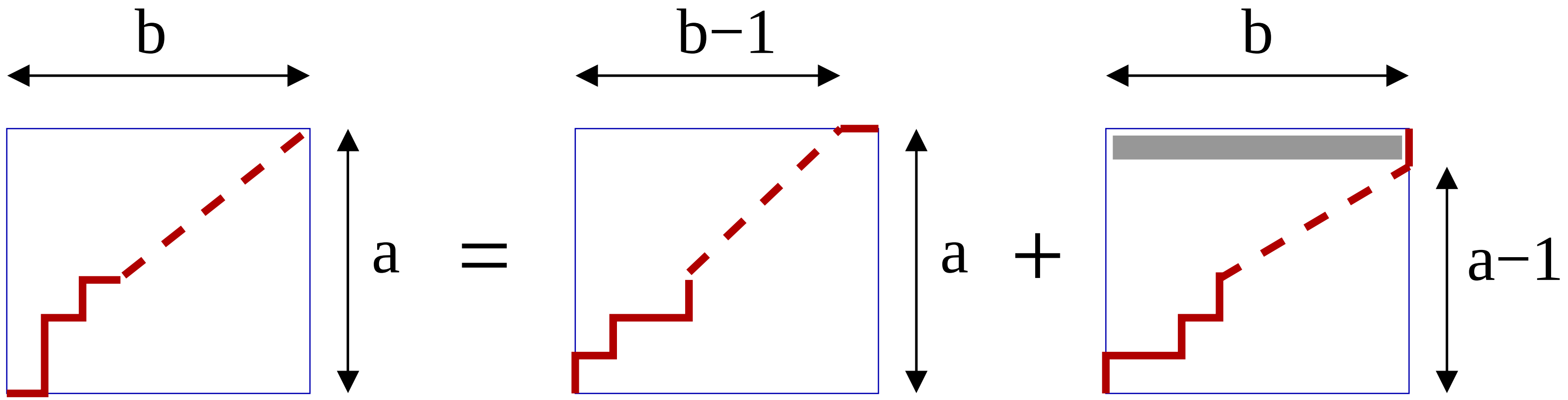
Example: $a = 2, b = 3$



$$\begin{bmatrix} 2+3 \\ 2,3 \end{bmatrix}_q = 1 + q + 2q^2 + 2q^3 + 2q^4 + q^5 + q^6.$$

Proof

$$\begin{bmatrix} a+b \\ a,b \end{bmatrix}_q = \begin{bmatrix} a+b-1 \\ a,b-1 \end{bmatrix}_q + q^b \begin{bmatrix} a+b-1 \\ a-1,b \end{bmatrix}_q.$$

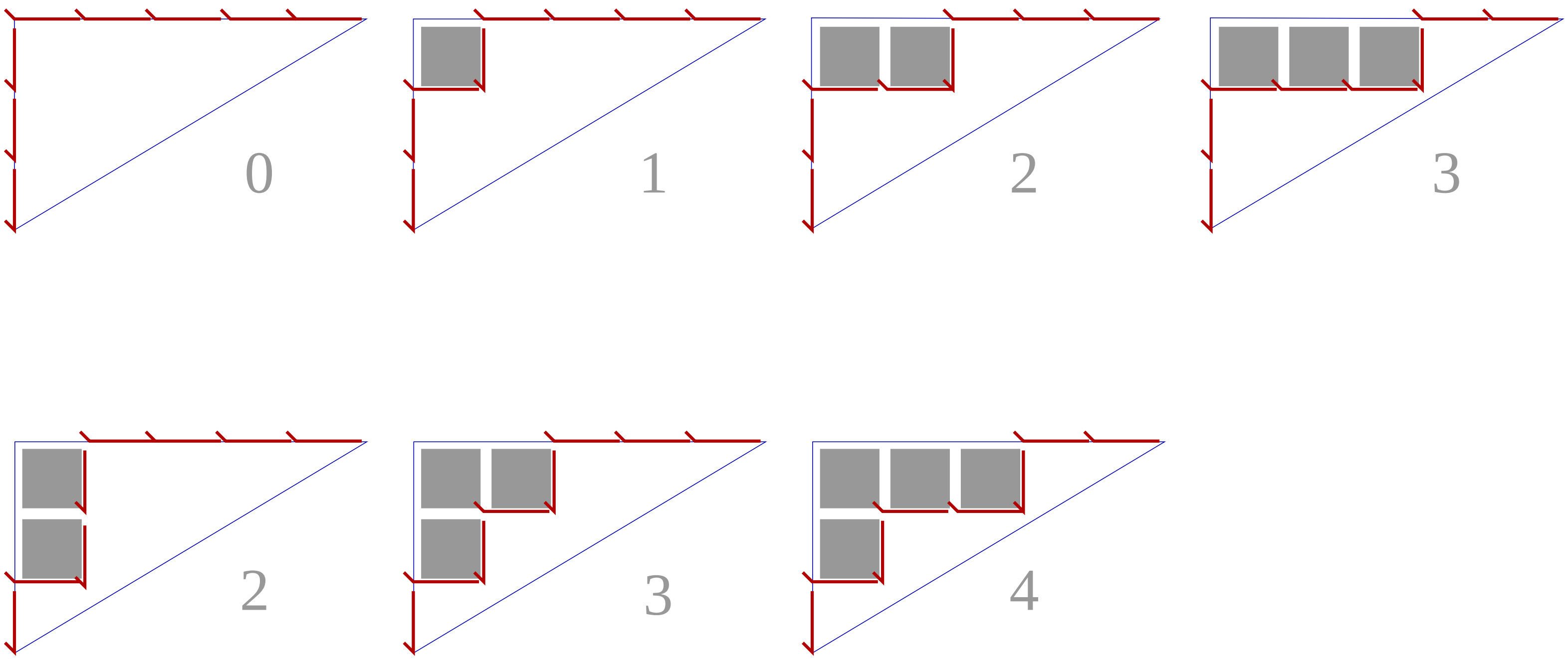


q, t-Catalan

$$\mathsf{Cat}_{a,b}(q) = \frac{1}{[a+b]_q}\begin{bmatrix} a+b \\ a,b \end{bmatrix}_q.$$

$$\begin{aligned}\mathsf{Cat}_{3,5}(q) &= \frac{1}{[8]_q}\frac{[8]_q[7]_q[6]_q}{[3]_q[2]_q[1]_q} \\ &= \frac{1-q^7}{1-q}\frac{1-q^6}{1-q}\frac{1-q}{1-q^3}\frac{1-q}{1-q^2} \\ &= 1+q^2+q^3+q^4+q^5+q^6+q^8.\end{aligned}$$

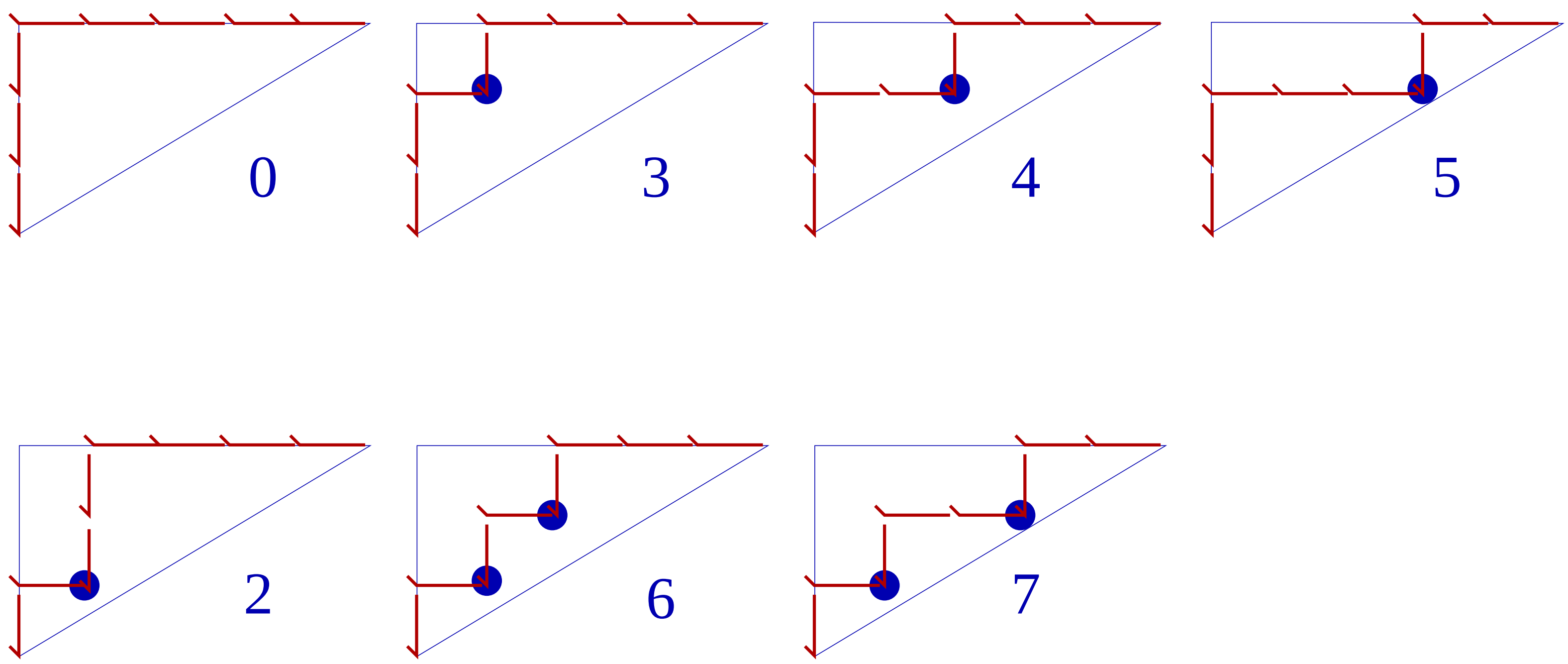
$|\mu|$?



$$\text{Above} = 1 + q + 2q^q + 2q^3 + q^4,$$

$$\mathbf{Cat}_{3,5}(q) = 1 + q^2 + q^3 + q^4 + q^5 + q^6 + q^8.$$

maj?



$$\text{Above} = 1 + q^2 + q^3 + q^4 + q^5 + q^6 + q^7,$$

$$\textbf{Cat}_{3,5}(q) = 1 + q^2 + q^3 + q^4 + q^5 + q^6 + q^8.$$

Inspiration: the q, t-Catalan

Garsia and Haiman's rational functions $OC_n(q,t)$ satisfy

$$q^{\binom{n}{2}} OC_n(q, 1/q) = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n, n \end{bmatrix}_q .$$

So let's evaluate the rational q, t-Catalan at $t = 1/q$.

Definitions

Let $A_{\text{max}} = (a-1)(b-1)/2$.

The statistic $h^+_{b,a}(\mu)$ is defined to be

$$|\{c \in \mu :\ -a < a \cdot \textbf{arm}(c) - b \cdot \textbf{leg}(c) \leq b\}|.$$

Fact: $A_{\text{max}} - |\,\mathrm{sw}(\mu)| = h^+_{b,a}(\mu)$.

Rational q,t-Catalan

$$C_{a,b}(q,t) = \sum_{w \in \mathcal{D}^{\mathrm{path}}(N^a E^b)} q^{\text{area}(\mathrm{sw}(w))} t^{\text{area}(w)}$$

$$= \sum_{\mu \in \mathcal{D}^{\mathrm{ptn}}(a,b)} q^{A_{\max} - |\,\mathrm{sw}(\mu)|} t^{A_{\max} - |\mu|}$$

$$= \sum_{\mu \in \mathcal{D}^{\mathrm{ptn}}(a,b)} q^{h^+_{b,a}(\mu)} t^{A_{\max} - |\mu|}.$$

So

$$q^{A_{\max}} C_{a,b}(q, 1/q) = \sum_{\mu \in \mathcal{D}^{\mathrm{ptn}}(a,b)} q^{h^+_{b,a}(\mu) + |\mu|}$$

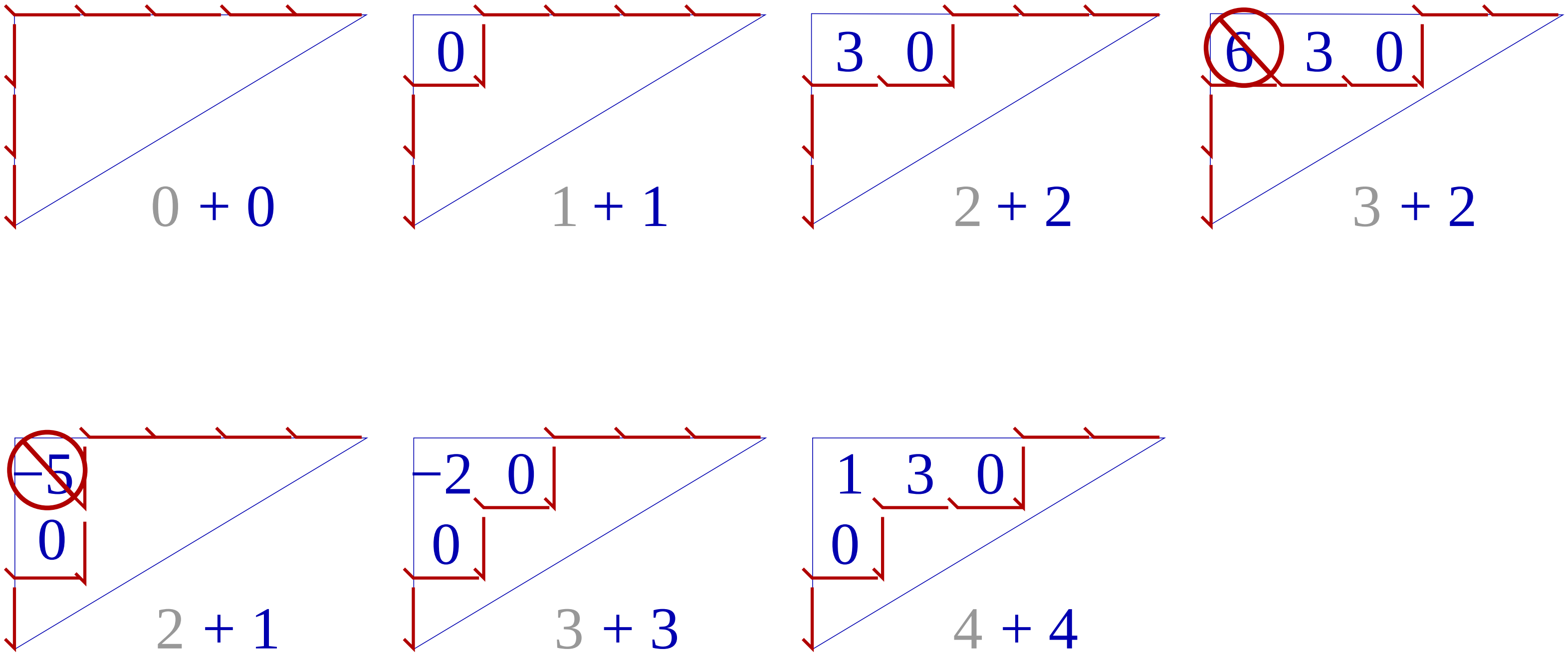
hopefully equals $\mathbf{Cat}_{a,b}(q)$.

q-Catalan Conjecture

For $\gcd(a,b) = 1$,

$$\mathsf{Cat}_{a,b}(q) = \sum_{\mu \in \mathcal{D}^{\mathrm{ptn}}(a,b)} q^{|\mu| + h^{+}_{b,a}(\mu)}.$$

Example: $a = 3, b = 5$

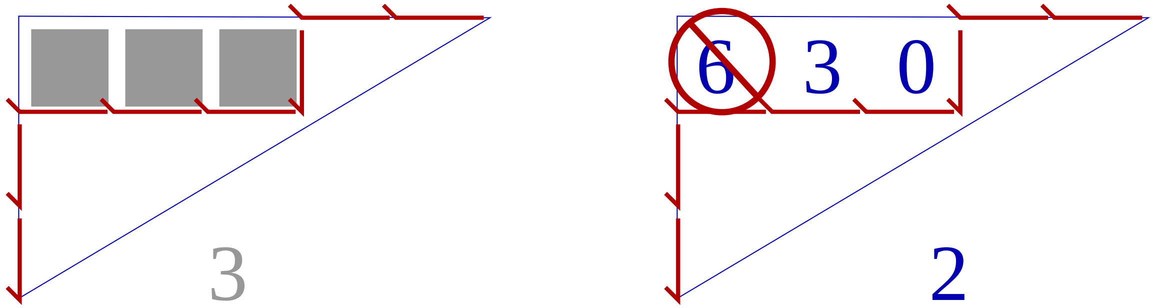


Above $= 1 + q^2 + q^3 + q^4 + q^5 + q^6 + q^8,$

Cat$_{3,5}(q) = 1 + q^2 + q^3 + q^4 + q^5 + q^6 + q^8.$

Yuck

The same path



contributes q^3 to $\begin{bmatrix} \mathbf{3+5} \\ \mathbf{3,5} \end{bmatrix}_q$ but

contributes q^{3+2} to $\mathbf{Cat}_{3,5}(q)$.

Reconciliation

For $\gcd(a,b)=1$,

$$\mathsf{Cat}_{a,b}(q) = \sum_{\mu \in \mathcal{D}^{\mathrm{ptn}}(a,b)} q^{|\mu| + h^{+}_{b,a}(\mu)}.$$

For $a, b \in \mathbb{N}$, can we find a mysterious, lucky statistic such that

$$\begin{bmatrix} a+b \\ a,b \end{bmatrix}_q = \sum_{\mu \in \mathcal{R}^{\mathrm{ptn}}(a,b)} q^{|\mu| + h^{+}_{b,a}(\mu) + \mathsf{ml}_{b,a}(\mu)}?$$

Tada!

Let $\mathbf{ml}_{b,a}(\mu) = -h^{+}_{b,a}(\mu)$. Then

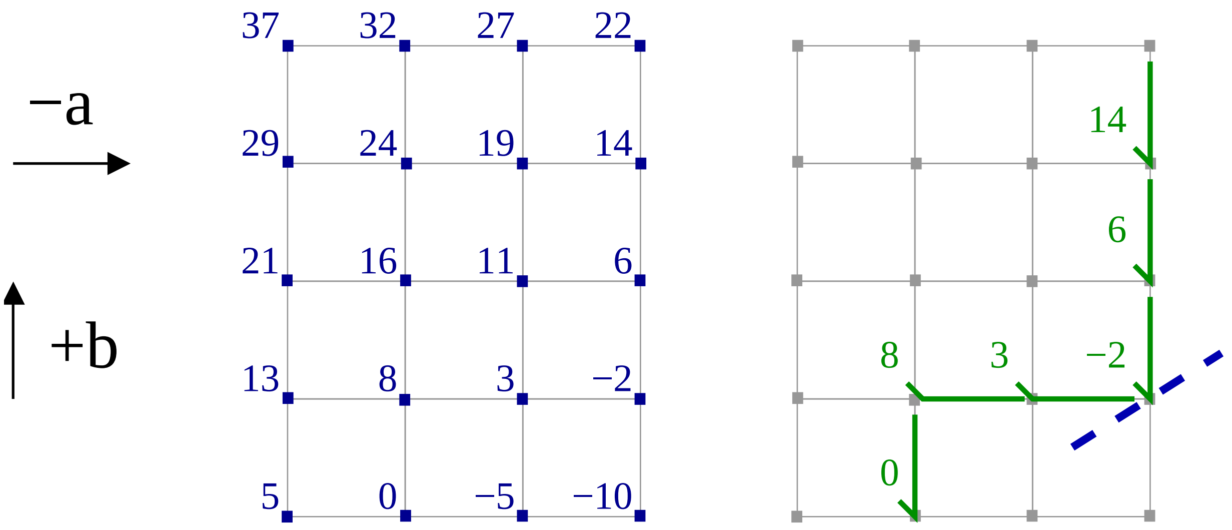
$$\begin{bmatrix} a+b \\ a,b \end{bmatrix}_q = \sum_{\mu \in \mathcal{R}^{\mathrm{ptn}}(a,b)} q^{|\mu| + h^{+}_{b,a}(\mu) + \mathbf{ml}_{b,a}(\mu)}$$
$$= \sum_{\mu \in \mathcal{R}^{\mathrm{ptn}}(a,b)} q^{|\mu|}.$$

So let's require $\mathbf{ml}_{b,a}(\mu) = 0$ when $\mu \in \mathcal{D}^{\mathrm{ptn}}(a,b)$.

Yes

The (a, b)-level of (x, y) is $ay - bx$.

Define $\mathbf{ml}_{b,a}(\mu)$ to be the minimum (a, b)-level of all points on the frontier of μ.

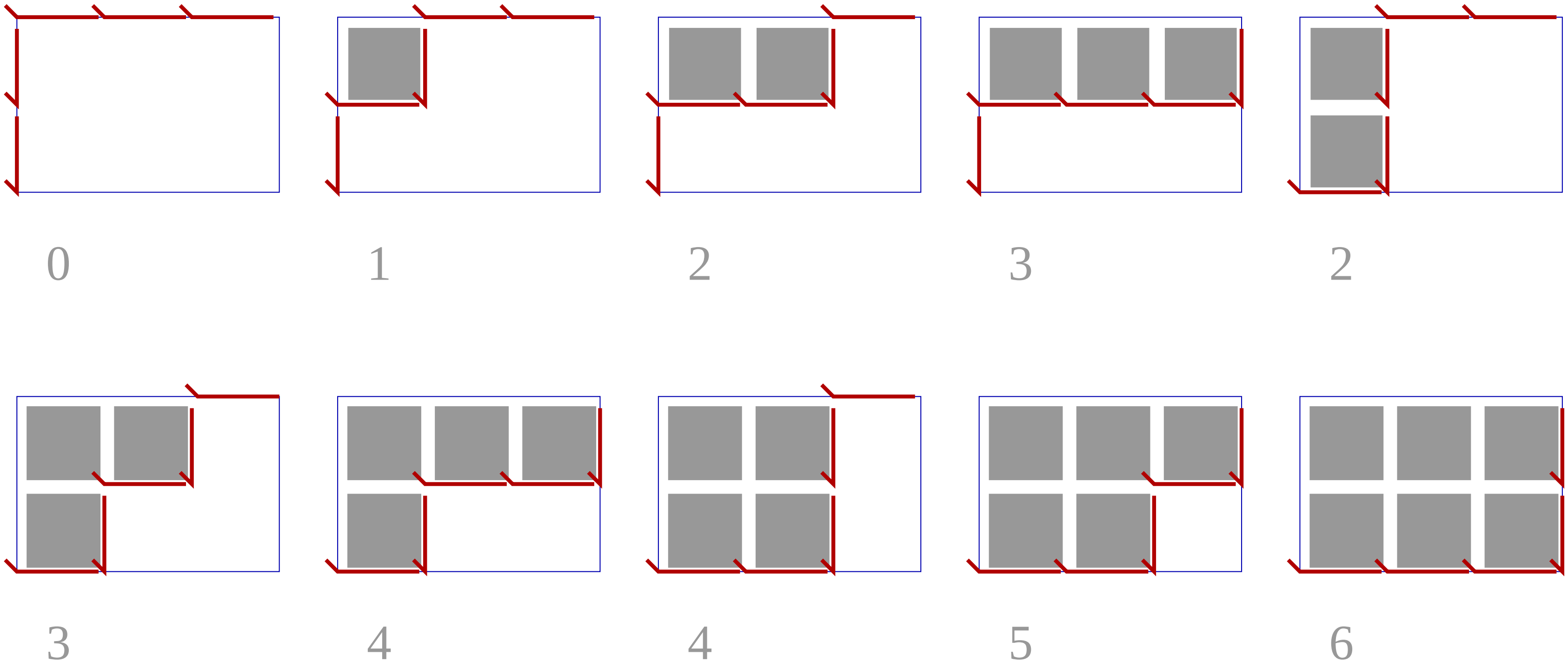


q-Catalan Conjecture

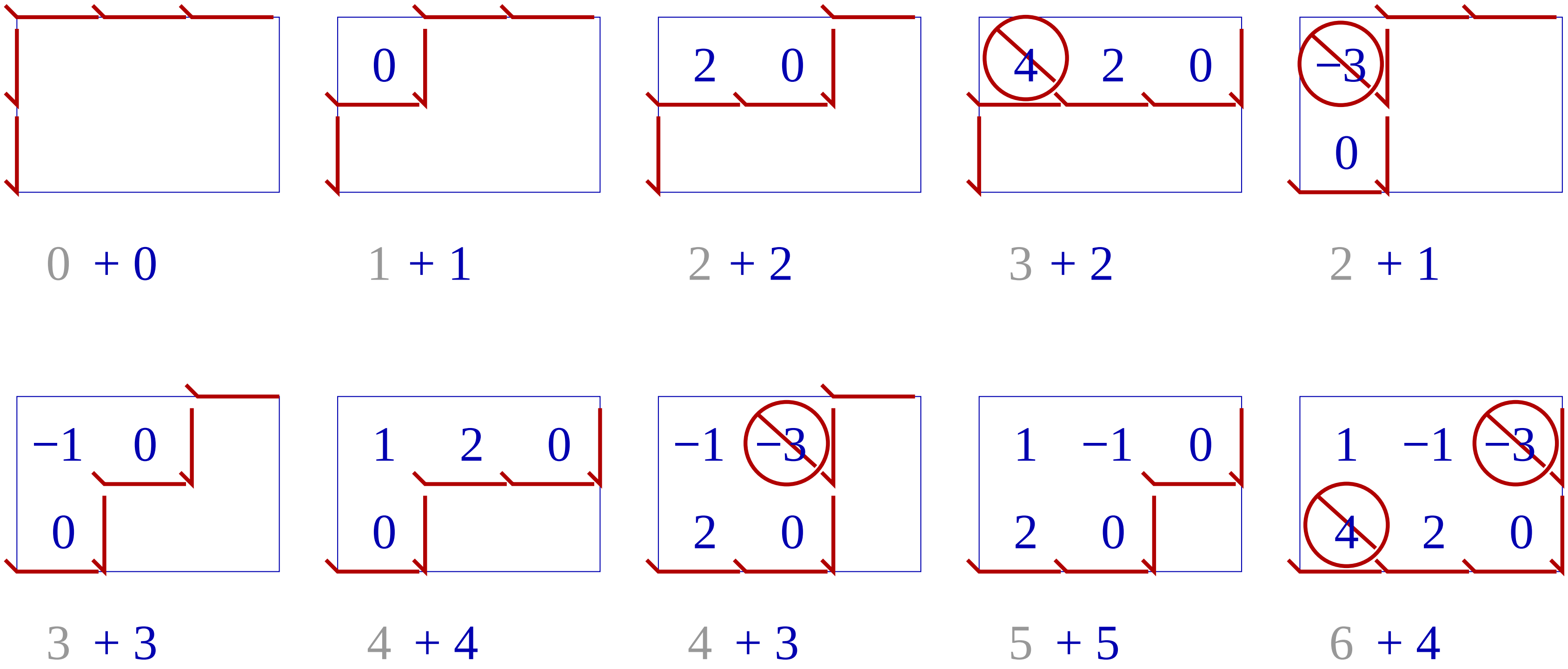
For all $\boldsymbol{a, b} \in \mathbb{Z}_{\geq 0}$, coprime or not,

$$\begin{bmatrix} \boldsymbol{a+b} \\ \boldsymbol{a, b} \end{bmatrix}_{\boldsymbol{q}} = \sum_{\boldsymbol{\mu} \in \mathcal{D}^{\mathbf{ptn}}(\boldsymbol{a,b})} \boldsymbol{q}^{|\boldsymbol{\mu}| + \mathbf{ml}_{\boldsymbol{b,a}}(\boldsymbol{\mu}) + \boldsymbol{h}^{+}_{\boldsymbol{b,a}}(\boldsymbol{\mu})}.$$

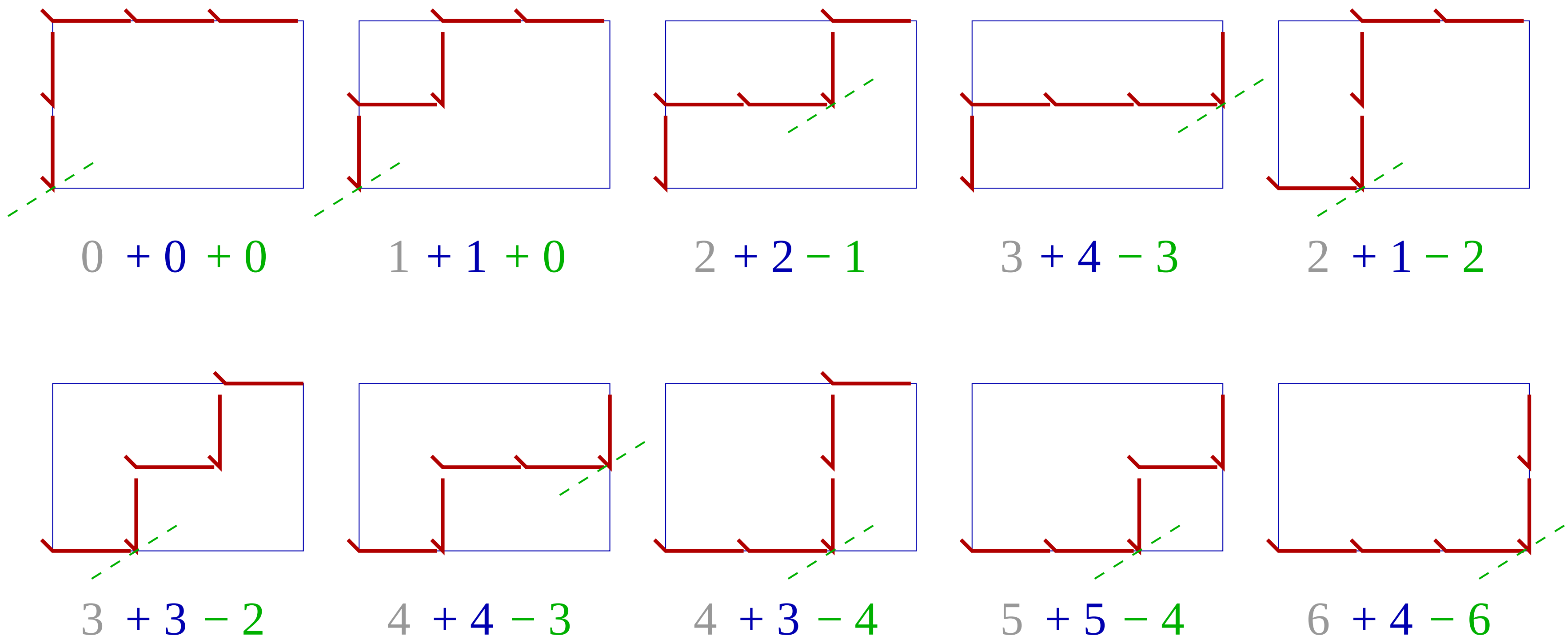
Example: $a = 2, b = 3$



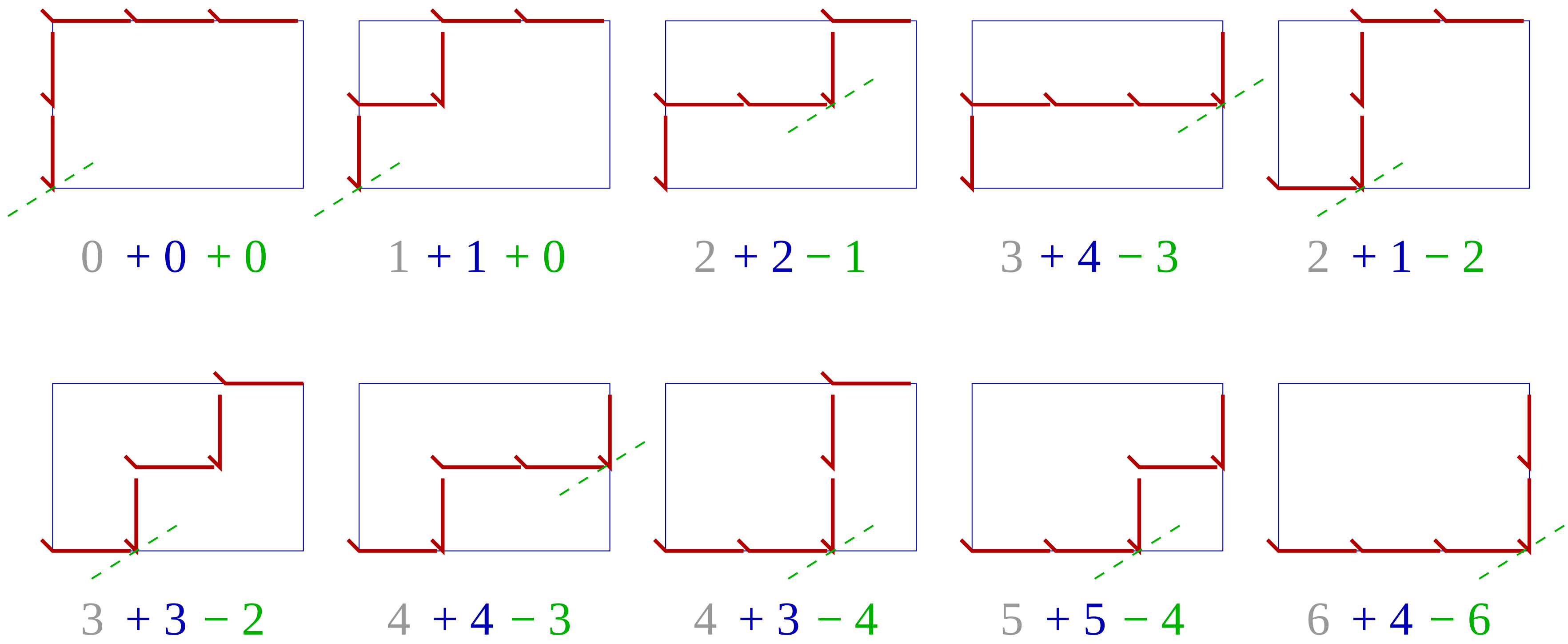
Example: $a = 2, b = 3$



Example: $a = 2, b = 3$



Example: $a = 2, b = 3$



$$1 + q + 2q^2 + 2q^3 + 2q^4 + q^5 + q^6$$

Proof?

Nope.

What we know

Theorem: For $\gcd(a, b) = 1$, the

q-Binomial Conjecture

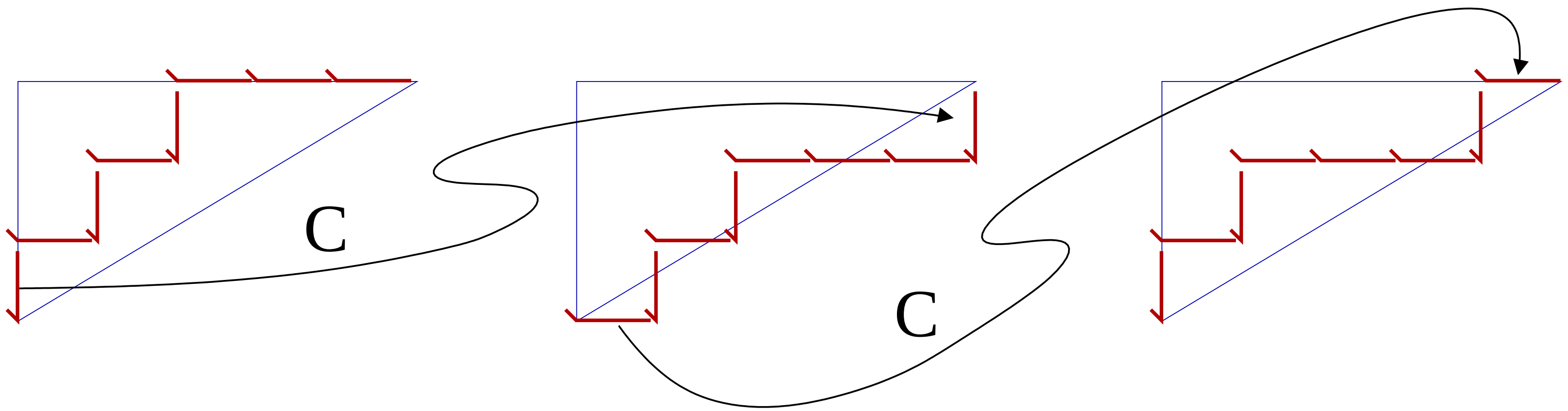
and the

q-Catalan Conjecture

are equivalent.

Sketchy proof

Define a cyclic shift map



Write $\mu^0, \mu^1, \ldots, \mu^{a+b-1}$ for the cyclic shifts of μ.

Claim 1

If ν is a cyclic shift of $\mu \in \mathcal{D}^{\mathrm{ptn}}(a,b)$, then

$$|\nu| + \mathbf{ml}_{b,a}(\nu) = |\mu|.$$

Sketchy proof

$$\begin{bmatrix} a+b \\ a,b \end{bmatrix}_q = \sum_{\mu^0 \in \mathcal{D}^{\mathrm{ptn}}(a,b)} \sum_i q^{|\mu^i| + \mathsf{ml}_{b,a}(\mu^i) + h^+_{b,a}(\mu^i)}$$

$$= \sum_{\mu^0 \in \mathcal{D}^{\mathrm{ptn}}(a,b)} \sum_i q^{|\mu^0| + h^+_{b,a}(\mu^i)}$$

$$= \sum_{\mu^0 \in \mathcal{D}^{\mathrm{ptn}}(a,b)} q^{|\mu^0|} \sum_i q^{h^+_{b,a}(\mu^i)}.$$

Claim 2

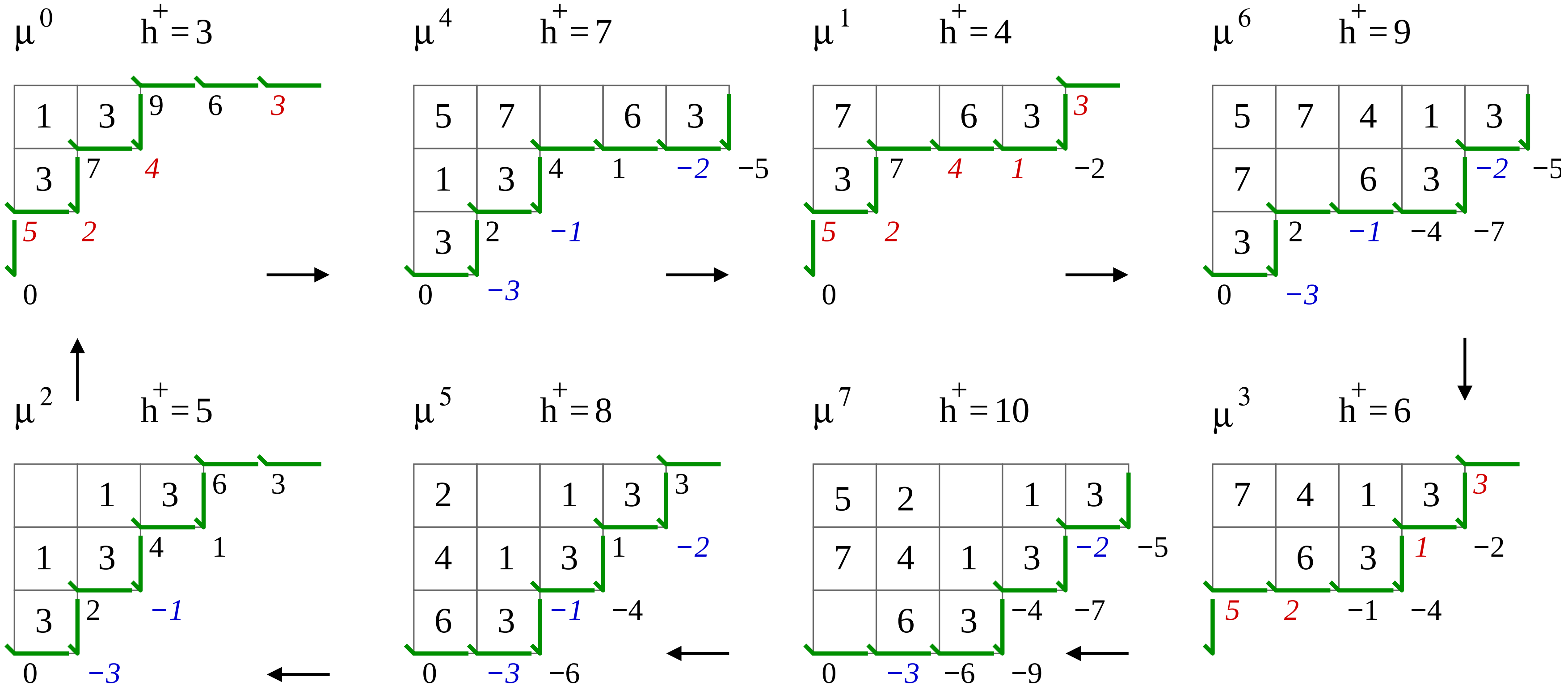
If the cyclic shifts of $\boldsymbol{\mu}^0 \in \mathcal{D}^{\mathrm{ptn}}(\boldsymbol{a}, \boldsymbol{b})$ are $\boldsymbol{\mu}^0, \ldots, \boldsymbol{\mu}^{a+b-1}$, then

$$\sum_i \boldsymbol{q}^{h^+_{b,a}(\boldsymbol{\mu}^i)} = [\boldsymbol{a} + \boldsymbol{b}]_q \boldsymbol{q}^{h^+_{b,a}(\boldsymbol{\mu}^0)}.$$

Sketchy proof

$$\begin{aligned}
\begin{bmatrix} a+b \\ a,b \end{bmatrix}_q &= \sum_{\mu^0 \in \mathcal{D}^{\mathrm{ptn}}(a,b)} \sum_i q^{|\mu^i| + \mathsf{ml}_{b,a}(\mu^i) + h^+_{b,a}(\mu^i)} \\
&= \sum_{\mu^0 \in \mathcal{D}^{\mathrm{ptn}}(a,b)} \sum_i q^{|\mu^0| + h^+_{b,a}(\mu^i)} \\
&= \sum_{\mu^0 \in \mathcal{D}^{\mathrm{ptn}}(a,b)} q^{|\mu^0|} \sum_i q^{h^+_{b,a}(\mu^i)} \\
&= \sum_{\mu^0 \in \mathcal{D}^{\mathrm{ptn}}(a,b)} q^{|\mu^0|} [a+b]_q q^{h^+_{b,a}(\mu^0)} \\
&= [a+b]_q \mathsf{Cat}_{a,b}(q).
\end{aligned}$$

Big example



μ 0
h+ = 3
1
3
3
9
6
3
7
4
5
2
0
μ 4
h+ = 7
5
7
6
3
1
3
3
4
1
−2
−5
2
−1
0
−3
μ 1
h+ = 4
7
6
3
3
3
7
4
1
−2
5
2
0
μ 6
h+ = 9
5
7
4
1
3
7
6
3
−2
−5
3
2
−1
−4
−7
0
−3
μ 2
h+ = 5
1
3
6
3
1
3
4
1
3
2
−1
0
−3
μ 5
h+ = 8
2
1
3
3
4
1
3
1
−2
6
3
−1
−4
0
−3
−6
μ 7
h+ = 10
5
2
1
3
7
4
1
3
−2
−5
6
3
−4
−7
0
−3
−6
−9
μ 3
h+ = 6
7
4
1
3
3
6
3
1
−2
5
2
−1
−4