Quasisymmetric expansions of Schur plethysms

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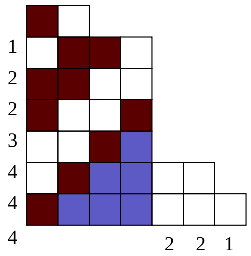

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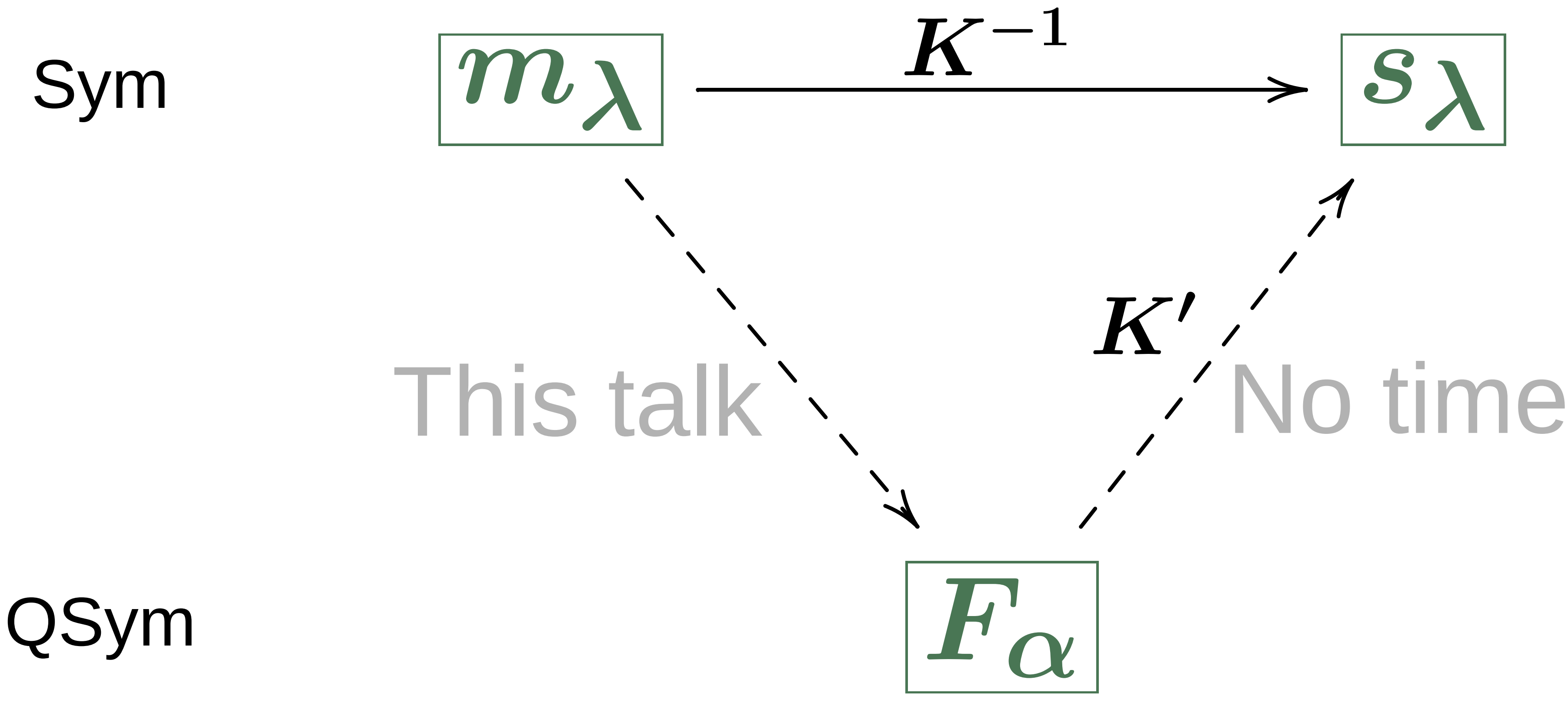
Expansions



Goal

$\boldsymbol{F}$-expansion of the plethysm $s_\mu[s_\lambda]$.

Warmup

$\boldsymbol{F}$-expansion of $\boldsymbol{s_\lambda}$

$$\boldsymbol{s_\lambda = \sum \ldots m_\mu,}$$ monomial SF

$$\boldsymbol{= \sum \ldots M_\beta,}$$ monomial quasi-SF

$$\boldsymbol{= \sum \ldots F_\alpha,}$$ fundamental quasi-SF

Schur functions

$$s_\lambda = \sum_{T \in SSYT(\lambda)} x^T$$

1	1	3
3	6	6
6		

$$\longrightarrow \quad x_1^2 x_3^2 x_6^3$$

Schur functions

$$s_\lambda = \sum_{T \in SSYT(\lambda)} x^T$$

1	1	3
3	6	6
6		

$\longrightarrow x_1^2 x_3^2 x_6^3$

Define $M_{223} = \sum_{i<j<k} x_i^2 x_j^2 x_k^3$

1	1
2	

1	1
3	

2	2
3	

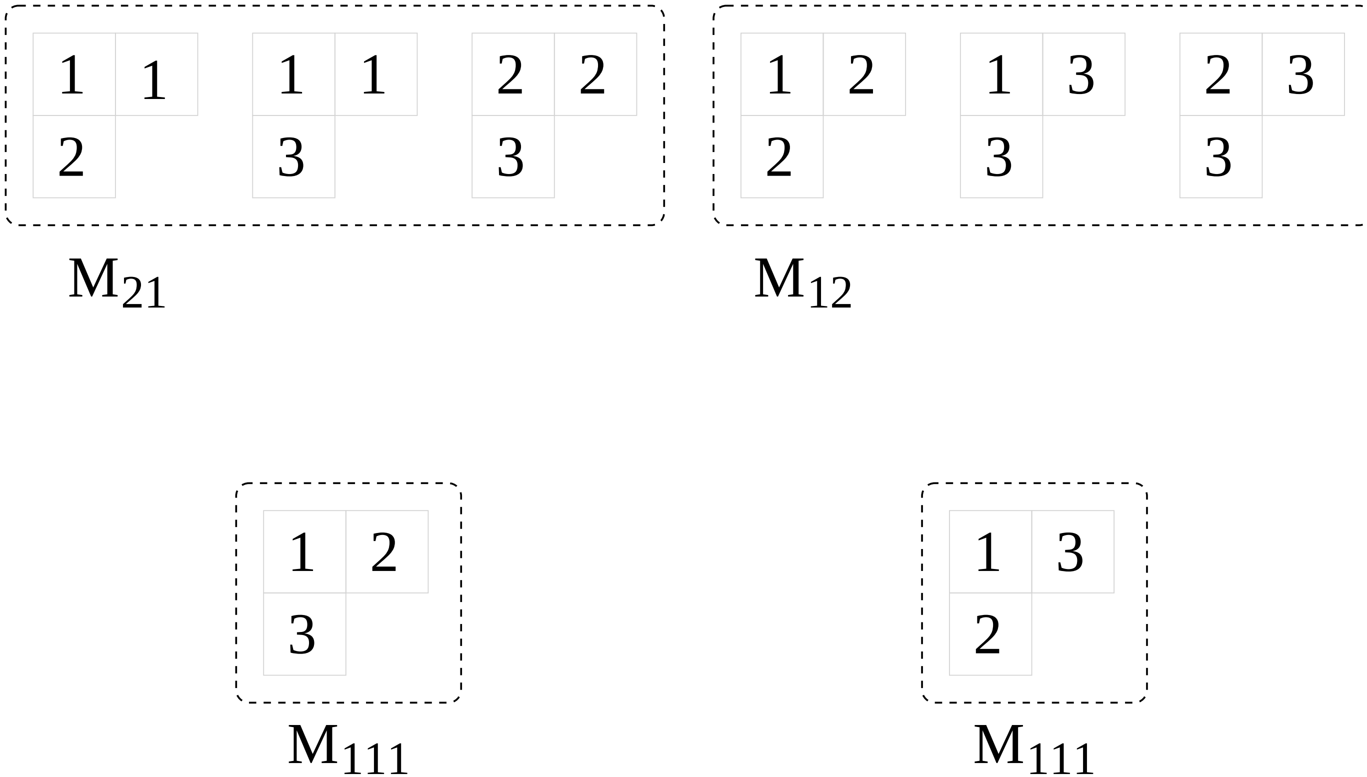
1	2
2	

1	3
3	

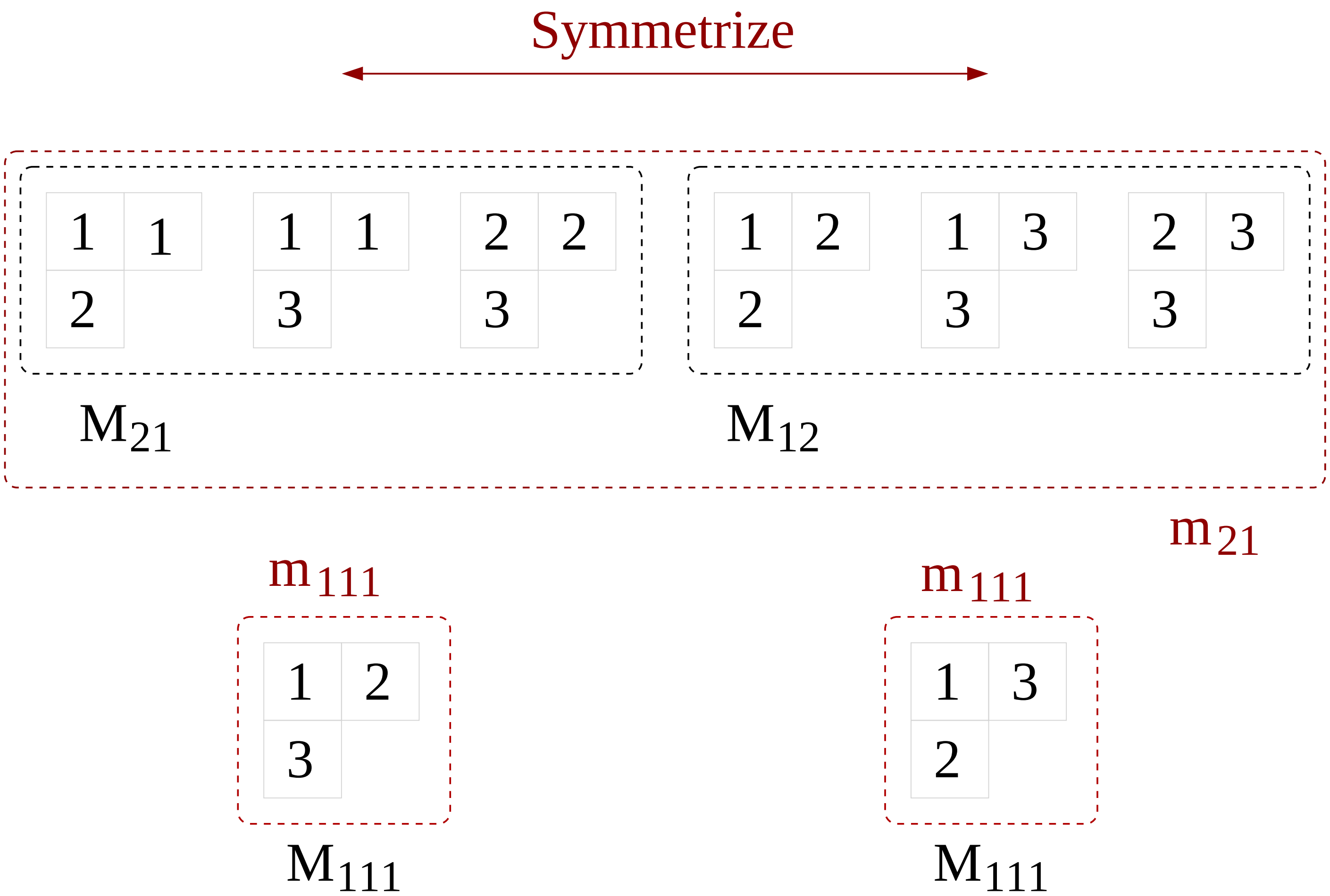
2	3
3	

1	2
3	

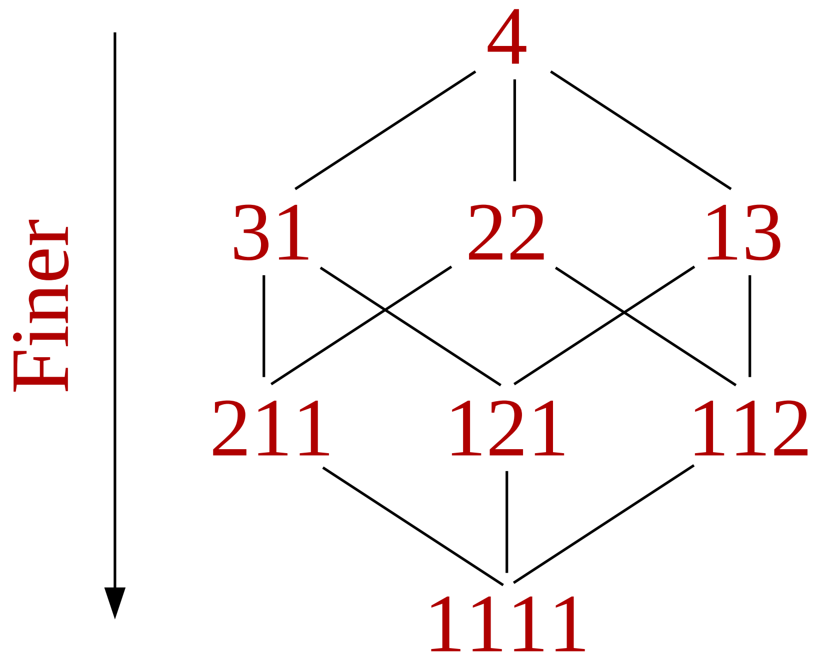
1	3
2	



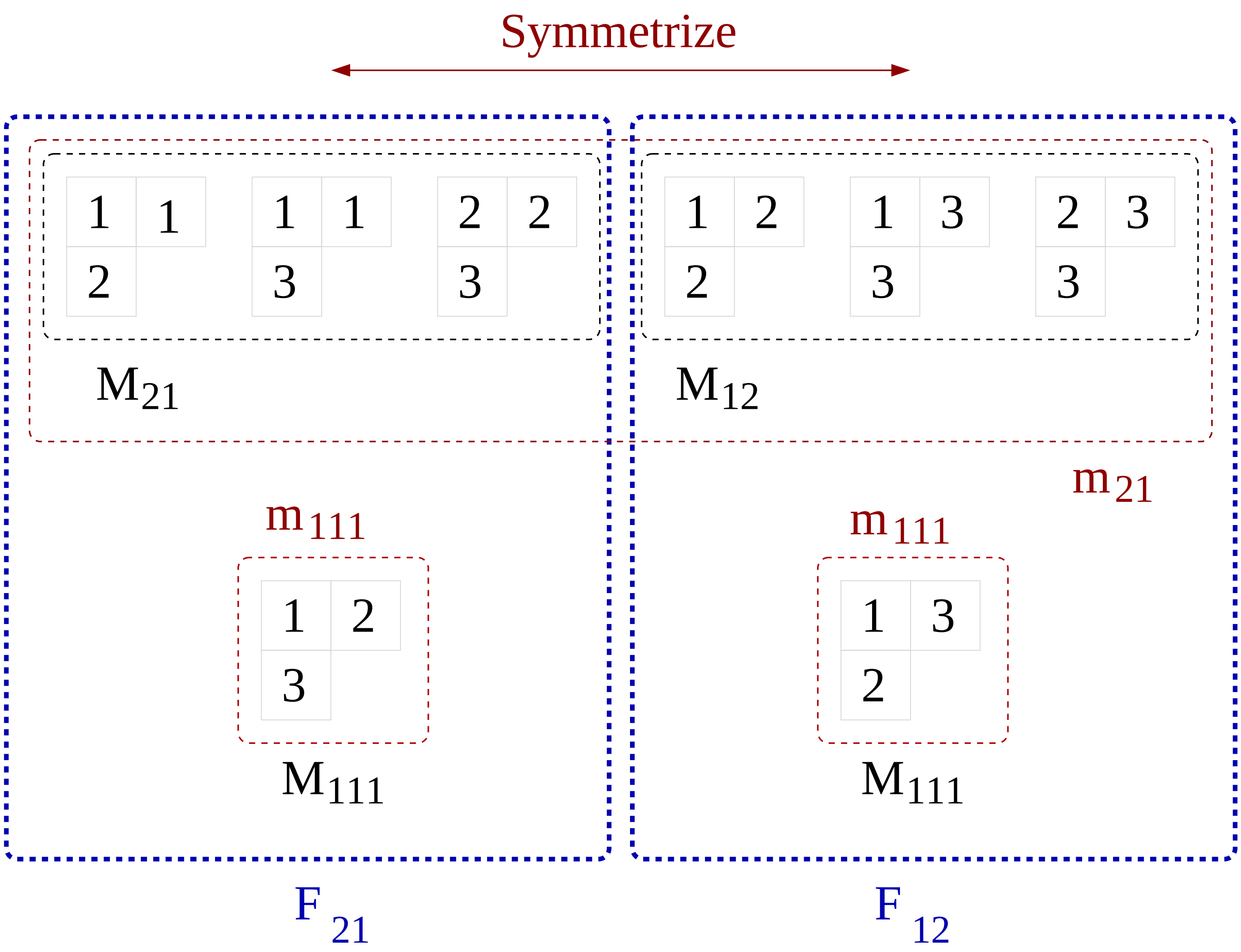
$$s_{21} = M_{21} + 2M_{111} + M_{12}$$



$$s_{21} = m_{21} + 2m_{111}$$



Define $F_\alpha = \sum_{\beta \text{ finer than } \alpha} M_\beta$



$$\boldsymbol{s_{21} = F_{21} + F_{12}}$$

Standardization

1	1	2	2
2	2	3	
3	3		

1	2	5	6
3	4	9	
7	8		

3 3 2 2 3 1 1 2 2

7 8 3 4 9 1 2 5 6

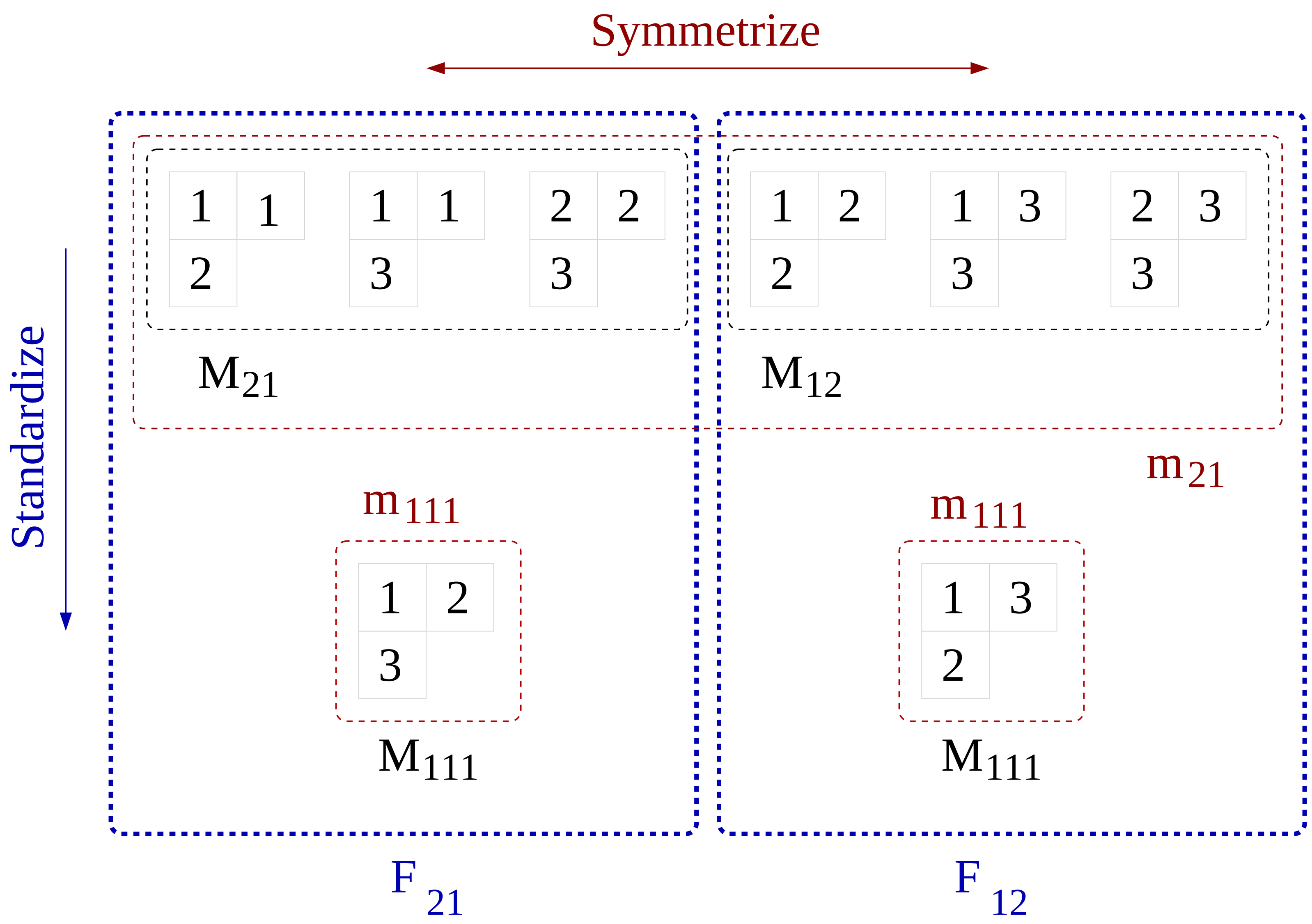
So $\mathbf{IDes} = \{\mathbf{2}, \mathbf{6}\} \to \mathbf{Asc} = \mathbf{243} \models \mathbf{9}$

Standardization

$$\alpha = 243 \quad \left(\begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 6 \\ \hline 3 & 4 & 9 \\ \cline{1-3} 7 & 8 \\ \cline{1-2} \end{array}, 112222333 \right) \leftrightarrow \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 2 \\ \hline 2 & 2 & 3 \\ \cline{1-3} 3 & 3 \\ \cline{1-2} \end{array}$$

$$\beta = 11313 \quad \left(\begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 6 \\ \hline 3 & 4 & 9 \\ \cline{1-3} 7 & 8 \\ \cline{1-2} \end{array}, 123334555 \right) \leftrightarrow \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 3 & 3 & 5 \\ \cline{1-3} 5 & 5 \\ \cline{1-2} \end{array}$$

$$\beta = 1^9 \quad \left(\begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 6 \\ \hline 3 & 4 & 9 \\ \cline{1-3} 7 & 8 \\ \cline{1-2} \end{array}, 123456789 \right) \leftrightarrow \begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 6 \\ \hline 3 & 4 & 9 \\ \cline{1-3} 7 & 8 \\ \cline{1-2} \end{array}$$



$$s_{21} = F_{21} + F_{12} = \sum_{T \in SYT(21)} F_{\mathrm{Asc}(T)}$$

Plethysm

Consider

- $x_1, \ldots, x_t$ variables
- $m_1, \ldots, m_s$ monic monomials in x_i
- $g(x_1, \ldots, x_t) = m_1 + \cdots + m_s$
- $f(x_1, \ldots, x_s)$ symmetric.

Define the plethysm

$$f[g] = f(m_1, m_2, \ldots, m_s).$$

Main result

Theorem[Loehr-W '10]

Let $\mu \vdash a$ and $\lambda \vdash b$. Then

$$s_\mu[s_\lambda] = \sum_{A \in \mathrm{Std}(\mu,\lambda)} F_{\mathrm{Asc}(A)}.$$

cf. Malvenuto-Reutenauer '98

Schur plethysms

Sample term in $h_3[h_5] = s_{(3)}[s_{(5)}]$

1 1 3 3 5	1 2 2 2 4	2 2 3 3 3

$\downarrow$

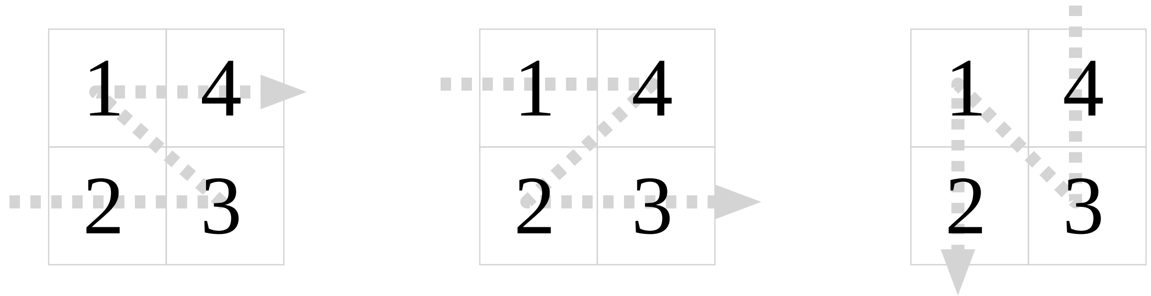
1	1	3	3	5
1	2	2	2	4
2	2	3	3	3

Q What is reading order?

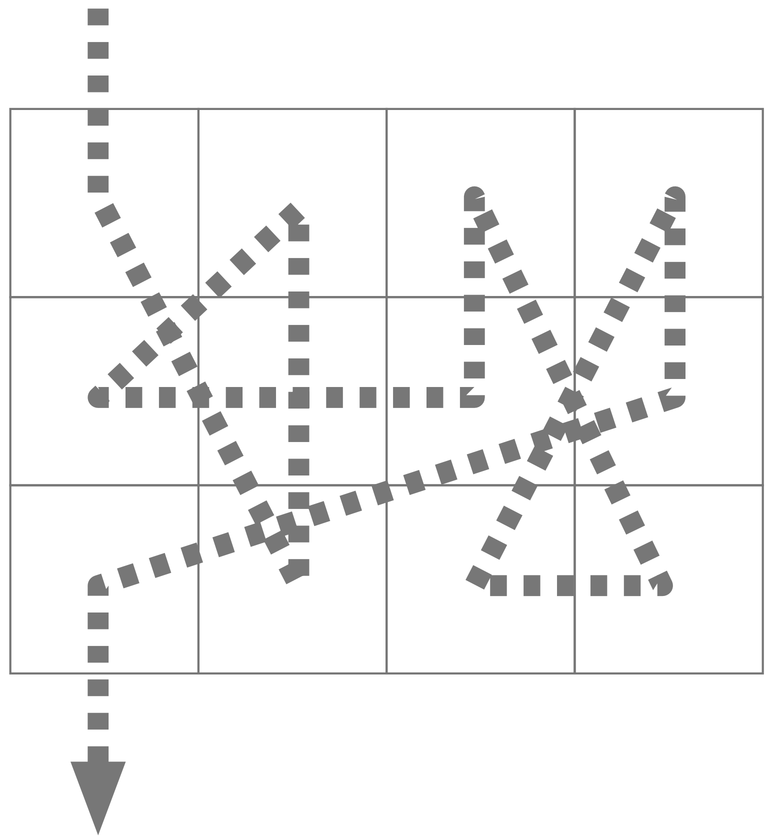
Naive approach

$$h_2[h_2] = F_4 + F_{22} + F_{121}$$

Order



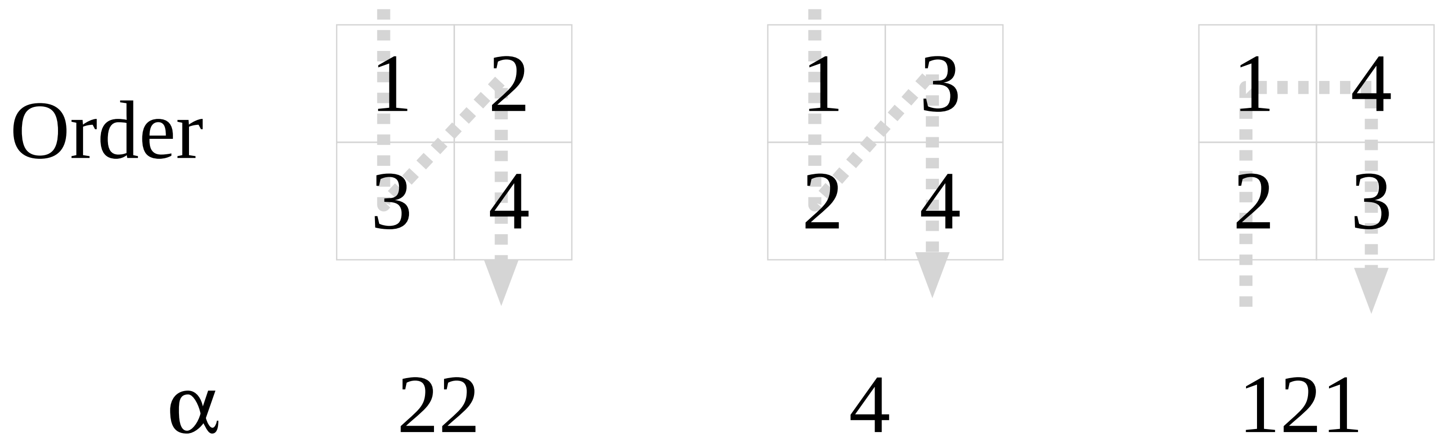
α	13	31	211



Correct approach

$$\boldsymbol{h_2[h_2] = F_4 + F_{22} + F_{121}}$$

The reading order is dynamic



Reading order for $h_a[h_b]$

1. Read columns from left to right.

2. Visit cells in i-th column according to order of cells in $(i+1)$-st column.

Which F_α do we live in?

A=

1	6	7	8
2	4	9	12
3	5	10	11

Which F_α do we live in?

A=

1	6	7	8
2	4	9	12
3	5	10	11

2 3 1

Which F_α do we live in?

A=

1	6	7	8
2	4	9	12
3	5	10	11

$\mathbf{2\ 3\ 1\ 6\ 4\ 5}$

Which F_α do we live in?

A=

1	6	7	8
2	4	9	12
3	5	10	11

2 3 1 6 4 5 7 10 9

Which F_α do we live in?

A=

1	6	7	8
2	4	9	12
3	5	10	11

$\mathbf{2\ 3\ 1\ 6\ 4\ 5\ 7\ 10\ 9\ 8\ 12\ 11}$

$\boldsymbol{A}$ contributes to $\boldsymbol{F}_{\mathbf{143121}}$